Frege's work

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Conceptual notation

Chapter 1-2.: Logic

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Judgment: a sentence can be asserted as a judgment, but it can occur as a part of a more complex sentence. In this latter case the part-sentence is not asserted.

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Basic judgments: logical truths (supported by semantic reasoning).

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1. A distinction must be made between the subjective and the objective, the psychological and the logical.
2. Never ask about the meaning of a word in isolation, but only in the context of the sentences.
3. Never forget the distinction between concept and object.

(The concept is the semantical value of a unary predicate)

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Frege's question: Are the units distinguishable or not?

The *Grundlagen*: lessons from the critical analysis

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But from this sequence of definitions, no answer follows to the question 'Is Julius Caesar a number?'.
(We didn't define numbers as objects. Julius Caesar problem.)

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Hume's principle formalized:

$$(Nx : F(x) = Nx : G(x)) \leftrightarrow$$
$$\exists b([1-1](b) \wedge \forall x(F(x) \rightarrow G(b(x))) \wedge$$
$$\forall y(G(y) \rightarrow \exists x(F(x) \wedge b(x) = y)))$$

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Fregean abstraction: We have an equivalence relation between some (concrete) objects and we can say that the equivalent objects share the common property.

We can also introduce abstract objects in this way. We assign the same abstract object to equivalent objects and different abstract objects to non-equivalent objects. An abstraction principle is the proposition that says that to two concrete objects belongs the same abstract object iff they are equivalent.

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Even sets can be introduced by abstraction in this way: the extension of two open sentences is the same set iff they are true for just the same objects (unlimited comprehension).

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Neo-Fregeanism: Let us introduce natural numbers simply by Hume's principle.