Frege numbers
Dedekind's numbers

András Máté

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This is roughly the same as saying that the number belonging to F is its equivalence class for equinumerosity.

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„Frege's theorem": Frege numbers satisfy the axioms of primitive Peano-arithmetics. I.e., 0 is not an immediate successor, $ISucc$ is one-to-one and mathematical induction holds.

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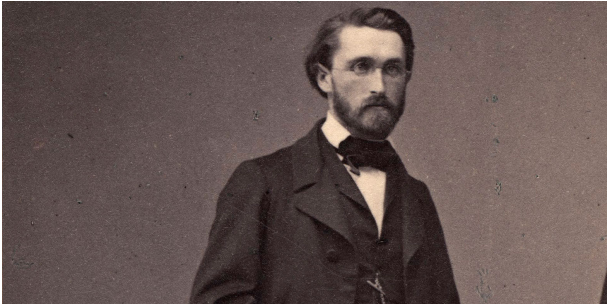
An introduction of abstract objects into a theory by an abstraction principle is a consistent extension of the theory *relative to set theory* if the equivalence classes generated by the principle are sets.

Richard Dedekind (1831-1916)



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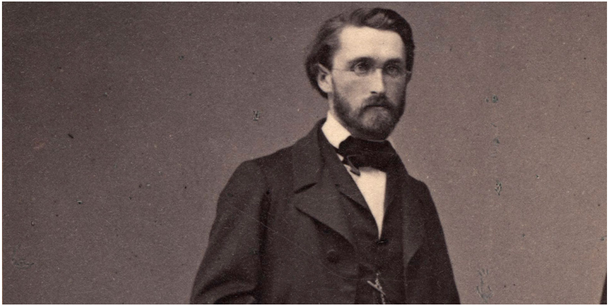

The grandfather of mathematical structuralism

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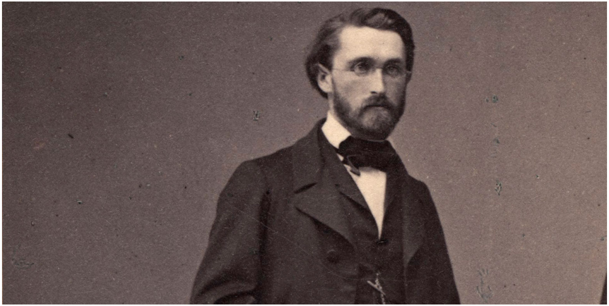

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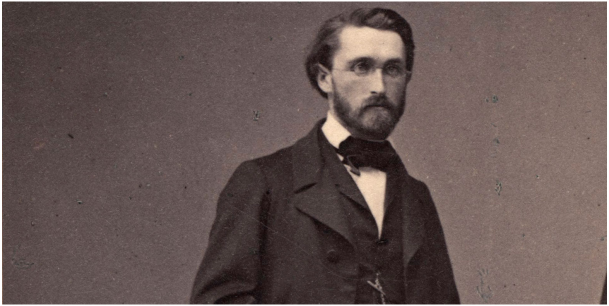

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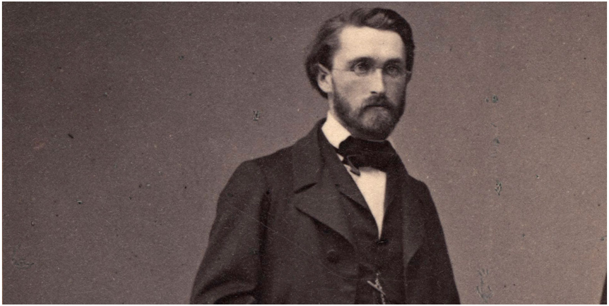

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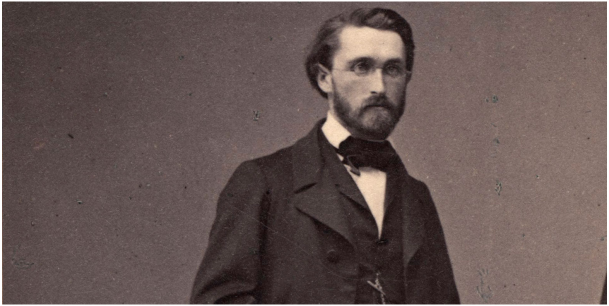

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3. William Lawvere's works on category theory (from the 1960's)

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Dedekind cut: Divide the rational numbers into two classes such that all members of the first (lower) class are less than any members of the second (upper) class. Such a classification is called cut.

There are three sorts of cuts:

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But what are the natural numbers?

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[A function φ is injective iff $\varphi(x) = \varphi(y) \rightarrow x = y$]

$S' = \varphi(S)$ is the system consisting of the φ-maps of the members of S. If φ is a similarity transformation, then it has a converse that is a similarity transformation again and φ is an one-to-one correspondence between the members of S and S'.

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If $A \subseteq S$, then the intersection of all chains containing A is a chain containing A and contained by S. It is the chain of A, A_0, or $\varphi_0(A)$.

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Theorem of complete induction: For any systems Σ and $A \subseteq \Sigma$, if for any $x \in A_0 \cap \Sigma$, $\varphi(x) \in A_0 \cap \Sigma$, then $A_0 \subseteq \Sigma$.

Infinity

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66. Theorem. There exist infinite systems.

Proof.* My own realm of thoughts, i. e., the totality S of all things, which can be objects of my thought, is infinite. For if s signifies an element of S, then is the thought s', that s can be object of my thought, itself an element of S. If we regard this as transform $\phi(s)$ of the element s then has the transformation ϕ of S, thus determined, the property that the transform S' is part of S; and S' is certainly proper part of S, because there are elements in S (e. g., my own ego) which are different from such thought s' and therefore are not contained in S'. Finally it is clear that if a, b are different elements of S, their transforms a', b' are also different, that therefore the transformation ϕ is a distinct (similar) transformation (26). Hence S is infinite, which was to be proved.

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Natural numbers: the elements of any simply infinite system N if we entirely neglect the special character of the elements; simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting transformation ϕ

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In other words, simply infinite systems are models of second order Peano arithmetics. The converse is also true: every model of second-order PA is a simply infinite system.

A negation-complete arithmetics ?!?

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Therefore, second-order Peano arithmetics (the set of *semantical* consequences of second-order Peano axioms) is negation complete.

Metalogical consequences

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Second-order logic cannot have a semantically complete calculus. Because if it had, then we could derive all semantic consequences from the second-order Peano axioms and obtain a negation complete axiomatic extension of first-order Peano arithmetics.

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What is arithmetical truth? The answer seems simple: a theorem of second-order PA. But the appearance of simplicity here is misleading.