A budget of paradoxes

The beginnings and aims of Hilbert's school

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We have proved a logical falsity from the (unlimited) comprehension using only logical rules.

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Suppose (for contradiction) that $H_0 = f(h)$.

$$h \in f(h) \leftrightarrow h \notin f(h)$$

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Because of the first conjunct in the scope of $\exists$, any concept F which makes the existential quantification true is true for the same objects as R (by Axiom V). Therefore, the right side is true iff $\neg R({}^{\vee}R)$.

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A collection of relevant paradoxes follows (with reference to De Morgan (1872) *A budget of paradoxes*).

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$$L \leftrightarrow \neg L$$

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$$L_S \leftrightarrow (L_S \text{ is not true})$$

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It is smaller than its successor.

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Known as Grelling-Nelson, Weyl, or simply heterological-paradox.

Two more famous paradoxes

The smallest number that cannot be defined in English in less than 81 characters

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where $d_n = 6$ if the nth digit after the decimal point of of a_n is 5 and $d_n = 5$ otherwise.

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a differs from any member of our sequence, but it is defined.

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Students take the test on Wednesday and are surprised.

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G is an ordinary game between two players iff it ends in finitely many moves. H is the following hypergame: the first player chooses an ordinary game, and then they play it. Is H an ordinary game or not?

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Russell's principle prohibits self-reference. It seems sufficient to avoid the previous paradoxes.

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It is a liar-like, but infinitary paradox that does not violate the vicious circle principle and does not contain any sort of self-reference.

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3. Abandon the priority of logic in favor of a more reliable basis (intuitionism)

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Standard way to investigate theories in the metatheory: formalize them in first-order logic.

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Hilbert's school: young mathematicians working in the 1920's in Göttingen on this program (Wilhelm Ackermann, Paul Bernays, John von Neumann, Jacques Herbrand).