# Formalism, Hilbert's program

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Propositions can have meaning, they can make true or false statements about some objects, but this is irrelevant to mathematics.

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Bernays (1928): 'Making us methodologically free from the intuition of space is not the same as ignoring the fact that the starting points of geometry lie in the intuition of space.'

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Reduction to logic cannot guarantee consistency. (This is the lesson of the paradoxes.)

# The program

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All such questions of principle . . . [sc. completeness, consistency, decidability] seem to me to constitute an important new field of research that still needs to be developed. In order to conquer this field we must . . . make the very notion of a specifically mathematical proof itself the object of investigation, just as . . . the physicist studies the theory of his apparatus, and the philosopher criticizes reason itself.

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The preparatory steps for such an investigation of a mathematical theory are the following:

- Axiomatize the theory
- Formalize the theory (including the logical principles used in it)

The main aim of the investigation is to prove that the risky, transfinite constituents don't make the theory inconsistent.

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$$\forall x A(x) \vee \exists x \neg A(x)$$

are trivially valid on a finite domain because they can be verified in finitely many steps. But on an infinite domain, after a finite number of steps, it is always possible that we have not find an object $a$ for which $\neg A(a)$ holds but we have not verified $\forall x A(x)$, either.

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Certainty does not lie in logic, but in experience and intuition (as the framework of experience).
Metamathematics is more reliable than other mathematical theories because it minimizes references to infinity.

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We can extend our system (consisting of real elements) with the ideal element $\omega$.

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'No one will drive us from the paradise which Cantor created for us.'

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With relative consistency proofs, we can reduce the problem of consistency of mathematical theories to the consistency of 'more fundamental' ones. The proofs must be purely formal and must not use anything other than the axioms.

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This first link could be a limited fragment of the arithmetics of natural numbers, with a limited logic (bounded quantifiers). In such a theory we would have to prove the consistency of the full Peano arithmetics, and then we could move forward.

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The risky component in arithmetics: mathematical induction. The induction scheme

$$(A(0) \wedge \forall x(A(x) \rightarrow A(x'))) \rightarrow \forall x A(x)$$

can only be used in cases where $A(x)$ contains no bounded variable.

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An overview of the results of Hilbert's school follows.

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$\Gamma$ is negation complete iff for every closed sentence $A$ of the language, either $A$ or $\neg A$ is in $Thm(\Gamma)$.

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A logical calculus is semantically complete iff all semantically valid inferences can be justified by derivation in the calculus.

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  - at least one formula of the form $A^{(a/x)}$;
  - every formula of the form $A^{(a/x)}$ where $a$ is any in-constant occurring in $\Gamma^*$;

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In case I, $\Gamma^*$ has a model (so $\Gamma$ has a model) whose domain consists only of natural numbers.

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  - no pair of sentences $A$, $\neg A$.

**Proposition** There is an algorithm that produces a sequence of closed sentences $\Gamma^*$ from $\Gamma$ s.t.:
Each step of the algorithm produces a consistent extension of $\Gamma$ and either

I $\Gamma^*$ is a finished analytic sequence for $\Gamma$
or

II $\Gamma^*$ is finite and contains a trivial contradiction.

In case I, $\Gamma^*$ has a model (so $\Gamma$ has a model) whose domain consists only of natural numbers.
In case II, $\Gamma$ is inconsistent.