Metatheorems about first-order logic

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28 March 2025

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In case I, Γ^* has a model (therefore Γ has a model, too) whose domain consists of natural numbers only.
In case II, Γ is inconsistent.

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Löwenheim-Skolem: If a set of sentences has a model, then it has a countable model, too.

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Not a contradiction; but it implies that some important notions (e.g. countability) are incurably relative, model-dependent. (Putnam: 'Models and reality', 1980)

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Every finite subset of this set has a model (namely the standard one extended by an appropriate interpretation of 'a'). Therefore, (due to compactness) the whole set has a model, too, and this is also a model of the axioms.

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BTW., nonstandard models of Peano arithmetics can be characterized by the following 2-order sentence:

$$\exists X(\exists x Xx \wedge \forall x(Xx \rightarrow x > 0) \wedge$$
$$\forall y[\forall x(Xx \rightarrow x > y) \rightarrow \forall x(Xx \rightarrow x > y')])$$

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(provided that Peano-arithmetics is ω-consistent).

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Second Incompleteness Theorem: The sentence expressing the consistency of Peano arithmetics is neither provable nor refutable (under the same conditions and with the same generalizations).

Kalmár's proof of the first incompleteness theorem

Language: first-order logic with 0 as an individual constant and some function symbols for arithmetic operations. Include at least the successor ($'$) and the four basic operations ($+$, $*$, $-$, $\div$).

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Numerals are the individual terms $0,\ 0',\ 0'',\ \ldots$.
Numerical terms are the terms containing no variable.
Assume that we can calculate the value of any numerical term.
Calculating a numerical term t means proving some equality $t = n$ (where n is a numeral).

A matrix of inequalities and its diagonal

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Consider the terms of the language that contain (at most) one free variable. These can be enumerated in an (infinite) sequence:

$$k_0(x),\ k_1(x),\ \ldots, k_n(x),\ \ldots$$

The indexes are the Gödel numbers of the terms.

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$$\begin{array}{lllll} k_0(x) \neq 0 & k_0(x) \neq 1 & \ldots & k_0(x) \neq n & \ldots \\ k_1(x) \neq 0 & k_1(x) \neq 1 & \ldots & k_1(x) \neq n & \ldots \\ \vdots & & & & \\ k_n(x) \neq 0 & k_n(x) \neq 1 & \ldots & k_n(x) \neq n & \ldots \end{array}$$

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Lemma (not proved): $f(x)$ can be expressed in our language by a term with one variable.

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If (G) is false, then for some n, $k_g(n) = f(n) = g$, and therefore P_n proves (G).
In summary, (G) is provable iff it is false.

The final result and some discussion

If our arithmetic (this could be Peano arithmetic or any effective extension of it) calculates every numerical term and *proves only true equalities with at most one variable*, then the Gödel sentence (G) is true and not provable, and its negation is not provable because it is false. Therefore it is negation incomplete.

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A plausible reading of the sentence (G): For every x, the value of the expression $k_g(x)$ (therefore $f(x)$) is different from g. This means that the diagonal formula numbered with g has no proof. But the g-th diagonal formula is (G) itself!! Therefore (G) says: 'I am not provable'.

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A consistent theory is ω-inconsistent iff there is some property P s.t. the theory proves $P(0)$, $P(1)$, ... $P(n)$, ... for each numeral n, but it proves $\exists x \neg P(x)$, too.

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PA + $\neg$ CPA is an example of a consistent, but ω-inconsistent theory (provided that Peano arithmetics is consistent).

Impact of the second incompleteness theorem

- Gödel: 'I wish to note expressly that [this theorem] does not contradict Hilbert's formalistic viewpoint. For this viewpoint presupposes only the existence of a consistency proof in which nothing but finitary means of proof is used and it is conceivable that there exist finitary proofs that *cannot* be expressed in the formalism of [first-order Peano arithmetics].' (Original paper on the incompleteness theorems)

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- von Neumann: 'Thus I am today of the opinion that
  1. Gödel has shown the unrealizability of Hilbert's program.
  2. There is no more reason to reject intuitionism (if one disregards the aesthetic issue, which in practice also for me be the decisive factor).'

  (Letter to Carnap, 1931)