The consistency of Peano arithmetics
Russell's logicism

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Our axioms, with the exception of induction axioms are verifiable formulas and that's all we need to know about them.

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  - The root is the closing formula of the deduction.
  - Each leaf is of one of the following sorts:
    1. Truths of propositional logic (tautologies)
    2. $\exists$-axioms: $A(t) \to \exists r A(r)$
    3. Equality formulas: $r = s \to (A(r) \to A(s))$
    4. Verifiable formulas
    5. Induction axioms

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$$A(0) \to \forall c(A(c) \to A(c')) \to A(a)$$

with an application of the following inference scheme (I):

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- ∃-scheme:

$$\frac{B(c) \to A}{\exists x B(x) \to A}$$

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By long and sometimes tricky calculation it turns out that we can transform our proof tree into a proof tree that deduces the closing formula from substitutions of verifiable formulas and tautologies (at the leafs) and uses only detachment as an inference rule.

The closing formula is deduced by this transformed tree from verified numerical equalities (substitutions of the axioms) using propositional logic only. Therefore, there is no reason to doubt it.

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- Elimination of I-inferences.The I-inference is only used to prove a truth about a specific number, such as 3. So we can replace it by inferences from 0 to 1, from 1 to 2, from 2 to 3.
- Elimination of forks. A fork is the following configuration in the proof tree: An existentially quantified formula is introduced somewhere using an ∃-scheme, and the same formula is the consequent of some ∃-axiom at some leaf. The idea is that the relevant existentially quantified formulas occur in such pairs. The paths from the two formulas to the closing formula must met at some node before the closing formula, otherwise the closing formula would contain a quantification. Forks can also be replaced by propositional logic proof trees.

What is remaining?

We still need to prove that from the proof of an arbitrary numerical formula *using a finite number of* iterated I-inference elimination and fork elimination is possible to obtain such a transformed proof tree. This is the part of our proof that cannot be formalized within 1-order Peano Arithmetic.

$0 - \omega$-figures

Recursive definition of the $\underline{0 - \omega\text{-figures}}$, with their ordering $<$ and classification into degrees:[1]

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Recursive definition of the $0 - \omega$-figures, with their ordering $<$ and classification into degrees:[1]

- The first and smallest figure is '0', the only member of degree 0.
- Members of the first degree are (non-empty) sum(expression)s of the form $\omega^0 + \omega^0 + ... + \omega^0$. The shorter is the smaller one, and 0 is smaller than any of them. Instead of ω^0, write 1, and instead of the sum the length r, write r.

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- Suppose we have already introduced the figures up to degree k with their ordering. An expression of the form

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belongs to the degree $k + 1$ iff

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  - or else iff it is a continuation of b.

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We prove that the elimination steps (strictly) decrease the ordinal.
The last step of our proof (next slides): we prove that there is no infinite decreasing sequence of our ordinals.
Therefore, the transformed tree can be achieved in finitely many steps.

Descending finite ordinals

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- It is enough to prove that every ordinal of the form ω^a (where a has the degree k) is descending finite.
- Let us have a decreasing sequence from ω^a. Its first member is $c = \omega^{a_1} + \omega^{a_2} + ... + \omega^{a_r}$, where $a_1 < a$. We should prove that c is descending finite.

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- c is not larger than $\omega^{a_1} + \omega^{a_1} + ... + \omega^{a_1}$ (shortly, $\omega^{a_1} \cdot r$). Therefore, if we have a descending chain from c, we can get a descending chain from $\omega^{a_1} \cdot r$ putting this latter ordinal to the beginning of the sequence. Therefore, if $\omega^{a_1} \cdot r$ is descending finite, then ω^a is descending finite, too.

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- Therefore we reduced the descending finiteness of ω^a to that of ω^{a_1}, where $a_1 < a$. Iterating this consideration, we can get a decreasing sequence $a > a_1 > a_2$... where the first member has the degree k and therefore by hypothesis the sequence is finite.

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- The last member must be of degree 0 or 1, otherwise the decreasing sequence could have continued. But the ordinals of degree 0 and 1 are descending finite.

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The 1-order Peano proofs (formalized as above) use two kinds of 'transfinite' tools: $\exists$-inferences and induction inferences. Our metalanguage proof has shown that both can be eliminated at the cost that the finiteness of the elimination procedure can only be proved by some stronger sort of induction.

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The 1-order Peano proofs (formalized as above) use two kinds of 'transfinite' tools: ∃-inferences and induction inferences. Our metalanguage proof has shown that both can be eliminated at the cost that the finiteness of the elimination procedure can only be proved by some stronger sort of induction.

BTW. we did not use the other transfinite tool (∃-inference or its equivalent existential instantiation) in the proof.

Russell's logicism

Russell's vicious circle principle (VCP):

> *„Whatever involves all of a collection must not be one of the collection;" or, conversely: „If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total."*

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It eliminates the Russell paradox, the Liar paradox, the paradox of the smallest number not definable by . . . letters, the Richard paradox, the hypergame paradoxes. It does not eliminate the Yablo paradox.

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It is impredicative because N belongs to the possible values of φ.

Russellian types

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The technical elaboration of predicativity is done through the theory of types.

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Not satisfactory because bound variables may occur in φ and according to VCP, they also should be of lower type than φ.

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- Bottom of the hierarchy: variables of type 0 ranging over a non-empty set of individuals.
- Type 1: classes of individuals. These can be defined by propositional functions $\varphi(x)$ containing variables of type 0.
- Type 2: classes of classes of individuals, etc.

Not satisfactory because bound variables may occur in φ and according to VCP, they also should be of lower type than φ.

Ramified theory of types: types are descending sequences of natural numbers.

Arithmetics in the theory of types

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defines natural numbers of the type t if the successor function maps type t into itself and φ belongs to a certain type higher than t. There is no impredicativity any more because N will belong to a higher type than φ.

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Problem: we cannot use the definition of number in our usual inductive proofs because the properties for which we want to use induction are of higher type than the type of φ.

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Russell and Whitehead, *Principia Mathematica* I-III. (1st edition: 1910, 11, 13).