Intuitionism – and a little overview

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Intuitionism:
Create a new mathematics based on intuitively clear notions and more rigorous methods of reasoning.

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Doubts about the existence of large sets.
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– "Funny" functions. Real analysis as an art of producing counterexamples.

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The anti-realist position: truth is not independent of our knowledge. It is created by us (at least in some respect). Julius König: A logic with LEM is God's logic because he is omniscient – it is not *our* logic.

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Michael Dummett (1925-2011): Abandons Brouwerian psychologism but keeps anti-realism. You understand the meaning of a mathematical proposition if you are able to recognize whether a construction is a proof of the proposition or not.

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Intuitionism: Mathematical objects (including truths, proofs, etc.) exist by construction. Objects that cannot be construct but can only be "proved" to exist – mainly through proofs by contradiction – should not be accepted as mathematical objects. The source of the paradoxes is that mathematics left the realm of the constructible.

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Brouwer: mathematical objects are created in the mind, by the two *acts of intuition*.

The first act

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"Completely separating mathematics from mathematical language and hence from the phenomena of language described by theoretical logic, recognizing that intuitionistic mathematics is an essentially languageless activity of the mind having its origin in the perception of a move of time. This perception of a move of time may be described as the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. If the twoity thus born is divested of all quality, it passes into the empty form of the common substratum of all twoities. And it is . . . this empty form, which is the fundamental intuition of mathematics."

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"[The] intuition of two-oneness creates not only the numbers one and two, but also all finite ordinal numbers, inasmuch as one of the elements of the two-oneness may be thought of as a new two-oneness, which process may be repeated indefinitely; this gives rise still further to the smallest infinite ordinal number ω."

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"Admitting two ways of creating new mathematical entities: firstly in the shape of more or less freely proceeding infinite sequences of mathematical entities previously acquired ...; secondly in the shape of mathematical species, i.e. properties supposable for mathematical entities previously acquired, satisfying the condition that if they hold for a certain mathematical entity, they also hold for all mathematical entities which have been defined to be 'equal' to it ...."

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The second act is to allow the creation of free choice sequences of mathematical objects previously created. This is the basis of the intuitionist continuum.

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"For this reason the intuitionist can never feel assured of the exactness of a mathematical theory by such guarantees as the proof of its being noncontradictory, the possibility of defining its concepts by a finite number of words. "

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This turn makes Ramsey's justifications for the axioms of choice and infinity inaccessible.

Mathematical existence: the formalist view

Formalism: mathematics has no special object, only a special method, formal deduction. Applicability and application are a matter outside mathematics, therefore the metaphysical nature of the objects of our theories is irrelevant to mathematics. "Existence in mathematics is nothing but consistency" (Hilbert); consistent (first-order) theories have models and that's all we need.

Brouwer's example of the opposition between the intuitionist and the formalist

"Let us now consider the concept: 'denumerably infinite ordinal number.' From the fact that this concept has a clear and well-defined meaning for both formalist and intuitionist, the former infers the right to create the 'set of all denumerably infinite ordinal numbers', the power of which he calls $\aleph_1$, a right not recognized by the intuitionist. Because it is possible to argue to the satisfaction of both formalist and intuitionist, first, that denumerably infinite sets of denumerably infinite ordinal numbers can be built up in various ways, and second, that for every such set it is possible to assign a denumerably infinite ordinal number, not belonging to this set, the formalist concludes: '$\aleph_1 > \aleph_0$ ',a proposition that has no meaning for the intuitionist."

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Brouwer about logic: "The . . . point of view that there are no non-experienced truths and that logic is not an absolutely reliable instrument to discover truths has found acceptance with regard to mathematics much later than with regard to practical life and to science."

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- Proof of $\exists x A(x)$: presenting a member d of the domain and proving $A(d)$.

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- Proof of $\neg A$: proof of $A \rightarrow \perp$.

B–**H**–**K** interpretation: Not a (formal) definition of the logical constants of intuitionistic logic, but just an informal descripition of their meaning because it is based on an informal notion of construction.

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$$A(x) \Longleftrightarrow_{def} \exists y \exists z (P(y) \wedge P(z) \wedge 2x = y + z)$$

is decidable again, therefore $\forall x(A(x) \vee \neg A(x))$ holds, too. But $\forall x A(x) \vee \neg \forall x A(x)$ does not hold because we don't know whether Goldbach's conjecture is true or not and therefore we are not in the position to assert either member of the disjunction.

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Another example: $B(x) \Longleftrightarrow_{def} \exists y(y > x \wedge P(y) \wedge P(y+2))$ is not a decidable predicate. Therefore $\forall x(B(x) \vee \neg B(x))$ does not hold.

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holds.
But indirect proof

$$((\neg A \to B) \wedge (\neg A \to \neg B)) \to A$$

is not generally valid.

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With predicate logic, the situation is a bit more difficult, but there is a negative translation function $\mathbf{g}$ from classical first-order logic (FOL) to intuitionist predicate logic s.t. for any first-order formula A, FOL proves $A \leftrightarrow \mathbf{g}(A)$, intuitionist predicate logic proves $\mathbf{g}(A) \leftrightarrow \neg\neg\mathbf{g}(A)$ and if FOL proves A, then intuitionist predicate logic proves $\mathbf{g}(A)$.

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Intuitionist logic has several different semantics. Perhaps the most important of these are the Kripke-structures, with soundness and completeness theorems. In the case of propositional logic: Kripke-structures are trees and nodes of a branch of a tree represent (roughly) successive states of research.