Intuitionism continued

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HA is capable of Gödelisation, therefore incompleteness theorems are valid for it.

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Intuitionist theory of real numbers is *incomparable* with classical real analysis. Some true propositions of classical analysis are not true intuitionistically, but there are theorems of intuitionist analysis which are not true classically.

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Be $A(n)$ a decidable predicate of natural numbers for which we don't know whether $\forall n A(n)$ is true or not; say, '$2n$ is the sum of two prime numbers'. Let us define a sequence of real numbers:

$$r_n = \begin{cases} 2^{-n} & \text{if } \forall m \leq n. A(m) \\ 2^{-m} & \text{if } \neg A(m) \wedge m \leq n \wedge \forall k < m. A(k) \end{cases}$$

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Therefore, the proposition $(r = 0) \vee (r \neq 0)$ does not hold.

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Every total real function is continuous.

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Let f be a continuous real-valued function on the interval $[a, b]$ such that $f(a) < 0 < f(b)$. Then there is a $c \in [a, b]$ for which $f(c) = 0$.

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In general, instead of existence theorems intuitionists often have theorems about the existence of approximations within arbitrary precision.

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This is unacceptable for the intuitionist. But there are weaker versions of AC which are acceptable (and important for classical mathematics, too): countable choice, dependent choice.