A historical introduction to the philosophy of mathematics

Fall Semester 2026

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General information

Webpage: https://lps.elte.hu/andras/matfil/matfil.html

Presentations will be posted after the classes.

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Closing and grades:

The course will conclude with a conference on the philosophy of mathematics.

To receive a grade, you must give a presentation on a topic related to the course theme.

During the lectures, I will mention some possible topics (but there are other options).

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This picture about mathematics may be called the dogmatic view.

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Figures or *calculi* (small stones used to represent numbers) are just auxiliary tools to help our reason.

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Mathematics is nothing else than advanced logic.

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Fundamental truths are based on the properties of human cognitive capacity.

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Comment by Bolzano: first, it must be proved that there is a number which is the sum of the series.

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Bolzano: the paradoxes of infinite(ly small or large) quantities can be resolved by scrupulous (re-)defining basic concepts and proving everything what seems to be obvious.

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His conclusion: The paradoxes of infinite manifolds cannot be solved at all.

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Theory of infinite sets, infinite cardinals and ordinals.

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Is the Cantorian concept of set as clear and well supported as it seems to be?

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18th century: several attempts to prove the axiom (typically by contradiction).

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About twenty years later the equiconsistency of Euclidean and Bolyai-Lobachevsky-geometry is proved (Cayley-Klein model). It is therefore impossible to decide which is the true science of the space.

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This is just another remaining problem.