Some canonical calculi and logical languages
The concept of hypercalculus

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The (primitive) logical constants of first-order logic are the usual ones. The alphabet of our first-order language:

$$\mathcal{A}_{Language(FOL)} = \{(,\ ),\ \iota,\ o,\ \mathfrak{x},\ \varphi,\ \pi,\ =,\ \neg,\ \supset,\ \forall\}$$

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Auxiliary letters (with intended meanings in brackets): I (index), A (arity), V (variable), N (name functor), P (predicate), T (term), F (formula). We use calculus variables as needed (not to be changed with object-language variables).

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6.	$Ax \to Iy \to xN\varphi xy$	n-ary name functors

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10.	$Ax \to xoNy \to Tz \to xNyz$	Application of name functors with at least one argument

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16^*.	$Fx \to x$	Release rule

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The $\mathcal{A}_{Language(FOL)}$-strings derivable in this calculus are just the wff's of our $Language(FOL)$. By changing the release rule and/or leaving off some rules we could define other syntactical categories (terms, atomic formulas, etc.) of the language.

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Homework: How to change $K_{Language(FOL)}$ to define the terms resp. atomic formulas of our language?

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Two informal remarks:
1. Hypercalculi are canonical calculi just as any other calculus. *We* read the strings they produce as rules, derivability relations or calculi.
2. The calculus deriving the code of any canonical calculus also derives the code of itself – an innocent case of self reference.

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So the the strings that represent calculi will consist of the characters of the following alphabet:

$$\mathcal{A}_{cc} = \{\alpha,\ \beta,\ \xi,\ \gg,\ *\}$$

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- I (index)
- L (Translation of a letter of $\mathbf{C}$)
- V (Translation of a $\mathbf{C}$-variable)
- W (Translation of a word, i.e. variable-free string)
- T (Translation of a term, i.e. string of letters and variables)
- R (Translation of a $\mathbf{C}$-rule)
- K (Translation of an arbitrary calculus $\mathbf{C}$)

The calculus $\mathbf{H}_1$ (beginning)

1. I
2. $Ix \to Ix\beta$
3. $Ix \to L\alpha x$
4. $Ix \to V\xi x$
5. W
6. $Wx \to Ly \to Wxy$
7. T
8. $Tx \to Ly \to Txy$
9. $Tx \to Vy \to Txy$

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10. $Tx \to Rx$
11. $Tx \to Ry \to Rx \gg y$
12. $Rx \to Kx$
13. $Kx \to Ry \to Kx * y$

13^* $Kx \to x$

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This calculus derives the translation of any calculus over any alphabet (including its own translation $\mathbf{h}_1$).

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In the above description of the intended meaning, I have dropped the phrase 'translation of'. But never forget that we speak here not about the letters, variables, etc. of our hypercalculus, but about the strings translating the letters etc. of the original calculus.

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14. $Lu \to uSuSySx$
15. $\gg S \gg SySx$
16. $Vx \to Iz \to x\beta zSx\beta zSySx$
17. $Vx \to Iz \to xSxSySx\beta z$
18. $Vx \to Wy \to ySxSySx$
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Base: The substitution of the variable x by the word y makes y from x (rule 18.) and leaves any other character – letters (14.), the arrow (15.), other variables (16.-17) – unchanged.

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Base: The substitution of the variable x by the word y makes y from x (rule 18.) and leaves any other character – letters (14.), the arrow (15.), other variables (16.-17) – unchanged.
Inductive rule (19.): If the substitution makes v from u and w from z, then from their concatenation uz it makes the concatenation of the results vw.

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20. $Rx \to xDx$
21. $Rx \to Ky \to y * xDx$
22. $Rx \to Ky \to x * yDx$
23. $Rx \to Ky \to Kz \to y * x * zDx$
24. $zDu \to vSuSySx \to zDv$
25. $xDy \to xDy \gg z \to xDz$

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The calculus $\mathbf{H}_2$ consisting of the rules 1-25 derives Ka, Wb and aDb iff a is the translation of some calculus $\mathbf{C}$, b is the translation of a word c of the alphabet of $\mathbf{C}$ and $\mathbf{C}$ derives c. We can't give suitable release rules here.