

Inductive definitions and canonical calculi

András Máte

26.09.2025

Inductive classes and some notation

Inductive classes and some notation

Inductive classes (of strings over an alphabet $\mathcal{C}$) are subclasses of $\mathcal{C}^\circ$ that have an inductive definition.

Inductive classes and some notation

Inductive classes (of strings over an alphabet $\mathcal{C}$) are subclasses of $\mathcal{C}^\circ$ that have an inductive definition.

$\mathcal{C}^\circ$ is an inductive class itself (trivial).

Inductive classes and some notation

Inductive classes (of strings over an alphabet $\mathcal{C}$) are subclasses of $\mathcal{C}^\circ$ that have an inductive definition.

$\mathcal{C}^\circ$ is an inductive class itself (trivial).

Our rules have the form

$$\lceil a_1, a_2, \dots, a_n \in F \Rightarrow b \in F \rceil$$

We can write them equivalently as

$$\lceil a_1 \in F \Rightarrow a_2 \in F \Rightarrow \dots \Rightarrow a_n \in F \Rightarrow b \in F \rceil$$

Inductive classes and some notation

Inductive classes (of strings over an alphabet $\mathcal{C}$) are subclasses of $\mathcal{C}^\circ$ that have an inductive definition.

$\mathcal{C}^\circ$ is an inductive class itself (trivial).

Our rules have the form

$$\lceil a_1, a_2, \dots, a_n \in F \Rightarrow b \in F \rceil$$

We can write them equivalently as

$$\lceil a_1 \in F \Rightarrow a_2 \in F \Rightarrow \dots \Rightarrow a_n \in F \Rightarrow b \in F \rceil$$

Let us conventionally omit the reference to the class to be defined $\lceil \in F \rceil$ and remember to this by using $\rightarrow$ instead of $\Rightarrow$. So our rules have now the form

$$\lceil a_1 \rightarrow a_2 \rightarrow \dots \rightarrow a_n \rightarrow b \rceil$$

An example: numbers divisible by 3, in dyadic notation

An example: numbers divisible by 3, in dyadic notation

Let us use the alphabet $\mathcal{A}_d = \{0, 1\}$. The strings are 0-1 sequences including the dyadic numerals.

An example: numbers divisible by 3, in dyadic notation

Let us use the alphabet $\mathcal{A}_d = \{0, 1\}$. The strings are 0-1 sequences including the dyadic numerals.

The first numbers divisible by 3 are 0, 11 and 110. They will give the base for our definition. We can put them into the definition as input-free rules.

An example: numbers divisible by 3, in dyadic notation

Let us use the alphabet $\mathcal{A}_d = \{0, 1\}$. The strings are 0-1 sequences including the dyadic numerals.

The first numbers divisible by 3 are 0, 11 and 110. They will give the base for our definition. We can put them into the definition as input-free rules.

If a number is divisible by 3 and its numeral ends with 00, (so the numeral is of the form $x00$), then the next number divisible by 3 will be $x11$. As a formal rule,

$$x00 \rightarrow x11$$

An example: numbers divisible by 3, in dyadic notation

Let us use the alphabet $\mathcal{A}_d = \{0, 1\}$. The strings are 0-1 sequences including the dyadic numerals.

The first numbers divisible by 3 are 0, 11 and 110. They will give the base for our definition. We can put them into the definition as input-free rules.

If a number is divisible by 3 and its numeral ends with 00, (so the numeral is of the form $x00$), then the next number divisible by 3 will be $x11$. As a formal rule,

$$x00 \rightarrow x11$$

If our number is $x01$, then the next number divisible by 3 will be $y00$, where y is the *follower* of x . We must now encode the relation of following in the rule. We use an auxiliary letter F to do this:

$$x01 \rightarrow xFy \rightarrow y00$$

Numbers divisible by 3, continuation

Numbers divisible by 3, continuation

Similarly, we need the rules $x10 \rightarrow xFy \rightarrow y01$ and $x11 \rightarrow xFy \rightarrow y10$.

Numbers divisible by 3, continuation

Similarly, we need the rules $x10 \rightarrow xFy \rightarrow y01$ and $x11 \rightarrow xFy \rightarrow y10$.

Let us define the relation F inductively, too. Base: $x0Fx1$, rule: $xFy \rightarrow x1Fy0$. For technical reasons, we need to add $1F10$ to the base.

Numbers divisible by 3, continuation

Similarly, we need the rules $x10 \rightarrow xFy \rightarrow y01$ and $x11 \rightarrow xFy \rightarrow y10$.

Let us define the relation F inductively, too. Base: $x0Fx1$, rule: $xFy \rightarrow x1Fy0$. For technical reasons, we need to add $1F10$ to the base.

Our definition has now the following form: (see next slide)

Numbers divisible by 3, continuation2

0

11

110

$x0Fx1$

$1F10$

$xFy \rightarrow x1Fy0$

$x00 \rightarrow x11$

$x01 \rightarrow xFy \rightarrow y00$

$x10 \rightarrow xFy \rightarrow y01$

$x11 \rightarrow xFy \rightarrow y10$

Numbers divisible by 3, continuation2

0

11

110

$x0Fx1$

$1F10$

$xFy \rightarrow x1Fy0$

$x00 \rightarrow x11$

$x01 \rightarrow xFy \rightarrow y00$

$x10 \rightarrow xFy \rightarrow y01$

$x11 \rightarrow xFy \rightarrow y10$

This is now of the sort (form) of inductive definitions we call canonical calculus.

Canonical calculus (formal definition)

Canonical calculus (formal definition)

Let $\mathcal{C}$ be a (finite) alphabet and $'\rightarrow' \notin \mathcal{C}$. $\mathcal{C}$ -rules are defined inductively as follows:

Canonical calculus (formal definition)

Let $\mathcal{C}$ be a (finite) alphabet and ' $\rightarrow$ ' $\notin \mathcal{C}$. $\mathcal{C}$ -rules are defined inductively as follows:

- (i) If $f \in \mathcal{C}^\circ$, then f is a $\mathcal{C}$ -rule.
- (ii) If r is a $\mathcal{C}$ -rule and $f \in \mathcal{C}^\circ$, then $\lceil f \rightarrow r \rceil$ is a $\mathcal{C}$ -rule.

Canonical calculus (formal definition)

Let $\mathcal{C}$ be a (finite) alphabet and ' $\rightarrow$ ' $\notin \mathcal{C}$. $\mathcal{C}$ -rules are defined inductively as follows:

- (i) If $f \in \mathcal{C}^\circ$, then f is a $\mathcal{C}$ -rule.
- (ii) If r is a $\mathcal{C}$ -rule and $f \in \mathcal{C}^\circ$, then $\lceil f \rightarrow r \rceil$ is a $\mathcal{C}$ -rule.

Let $\mathcal{C}$ and $\mathcal{V}$ alphabets s.t. ' $\rightarrow$ ' $\notin \mathcal{C} \cup \mathcal{V}$. A finite class K of $\mathcal{C} \cup \mathcal{V}$ -rules is called a canonical calculus over $\mathcal{C}$. The members of K are the rules of K and the members of $\mathcal{V}$ (if any) are the variables of K .

Strings derivable in a canonical calculus

Strings derivable in a canonical calculus

Let $\mathcal{C}$ be an alphabet and K a canonical calculus over $\mathcal{C}$. The relation $K \mapsto f$ (read: „ K derives f ” or „ f is derivable in K ”) is defined by induction:

Strings derivable in a canonical calculus

Let $\mathcal{C}$ be an alphabet and K a canonical calculus over $\mathcal{C}$. The relation $K \mapsto f$ (read: „ K derives f ” or „ f is derivable in K ”) is defined by induction:

$$(i) \ f \in K \Rightarrow K \mapsto f$$

Strings derivable in a canonical calculus

Let $\mathcal{C}$ be an alphabet and K a canonical calculus over $\mathcal{C}$. The relation $K \mapsto f$ (read: „ K derives f ” or „ f is derivable in K ”) is defined by induction:

- (i) $f \in K \Rightarrow K \mapsto f$
- (ii) If $K \mapsto f$ and f' is the result of substituting a $\mathcal{C}$ -string for all occurrences of a variable in f , then $K \mapsto f'$ (Substitution)

Strings derivable in a canonical calculus

Let $\mathcal{C}$ be an alphabet and K a canonical calculus over $\mathcal{C}$. The relation $K \mapsto f$ (read: „ K derives f ” or „ f is derivable in K ”) is defined by induction:

- (i) $f \in K \Rightarrow K \mapsto f$
- (ii) If $K \mapsto f$ and f' is the result of substituting a $\mathcal{C}$ -string for all occurrences of a variable in f , then $K \mapsto f'$ (Substitution)
- (iii) If $K \mapsto f$, $K \mapsto f \rightarrow g$ and ‘ $\rightarrow$ ’ does not occur in f , then $K \mapsto g$ (Detachment)

Inductive (sub)classes

Inductive (sub)classes

Let $\mathcal{A}$ be an alphabet. The class of strings F is an inductive subclass of $\mathcal{A}^\circ$ iff there exist $\mathcal{C}$ and K s.t.

- $\mathcal{C}$ is an alphabet and $\mathcal{A} \subseteq \mathcal{C}$;
- K is a canonical calculus over $\mathcal{C}$;
- $F = \{x : x \in \mathcal{A}^\circ \wedge K \mapsto x\}$.

Inductive (sub)classes

Let $\mathcal{A}$ be an alphabet. The class of strings F is an inductive subclass of $\mathcal{A}^\circ$ iff there exist $\mathcal{C}$ and K s.t.

- $\mathcal{C}$ is an alphabet and $\mathcal{A} \subseteq \mathcal{C}$;
- K is a canonical calculus over $\mathcal{C}$;
- $F = \{x : x \in \mathcal{A}^\circ \wedge K \mapsto x\}$.

The members of the class $\mathcal{B} = \mathcal{C} - \mathcal{A}$ are the auxiliary letters.

Inductive (sub)classes

Let $\mathcal{A}$ be an alphabet. The class of strings F is an inductive subclass of $\mathcal{A}^\circ$ iff there exist $\mathcal{C}$ and K s.t.

- $\mathcal{C}$ is an alphabet and $\mathcal{A} \subseteq \mathcal{C}$;
- K is a canonical calculus over $\mathcal{C}$;
- $F = \{x : x \in \mathcal{A}^\circ \wedge K \mapsto x\}$.

The members of the class $\mathcal{B} = \mathcal{C} - \mathcal{A}$ are the auxiliary letters.

Homework: If F and G are inductive subclasses of some string class $\mathcal{A}^\circ$, then $F \cup G$ and $F \cap G$ are inductive subclasses of it, too.

Inductive (sub)classes

Let $\mathcal{A}$ be an alphabet. The class of strings F is an inductive subclass of $\mathcal{A}^\circ$ iff there exist $\mathcal{C}$ and K s.t.

- $\mathcal{C}$ is an alphabet and $\mathcal{A} \subseteq \mathcal{C}$;
- K is a canonical calculus over $\mathcal{C}$;
- $F = \{x : x \in \mathcal{A}^\circ \wedge K \mapsto x\}$.

The members of the class $\mathcal{B} = \mathcal{C} - \mathcal{A}$ are the auxiliary letters.

Homework: If F and G are inductive subclasses of some string class $\mathcal{A}^\circ$, then $F \cup G$ and $F \cap G$ are inductive subclasses of it, too.

A convention about the use of auxiliary letters: We use them to express predicates of strings. If we want to use P to express a monadic predicate, we write it as a prefix: Px . If it is an n -adic predicate ($n \geq 2$), we write it infix, on the following way:
 $x_1Px_2P \dots Px_n$.

Some additional remarks

Some additional remarks

Let $\mathcal{A}^\circ$ be the class of all strings over an alphabet $\mathcal{A}$.

Some additional remarks

Let $\mathcal{A}^\circ$ be the class of all strings over an alphabet $\mathcal{A}$.

- The empty class $\emptyset$ is an inductive subclass of $\mathcal{A}^\circ$.

Some additional remarks

Let $\mathcal{A}^\circ$ be the class of all strings over an alphabet $\mathcal{A}$.

- The empty class $\emptyset$ is an inductive subclass of $\mathcal{A}^\circ$.
- $\mathcal{A}^\circ$ is an inductive subclass of itself.

Some additional remarks

Let $\mathcal{A}^\circ$ be the class of all strings over an alphabet $\mathcal{A}$.

- The empty class $\emptyset$ is an inductive subclass of $\mathcal{A}^\circ$.
- $\mathcal{A}^\circ$ is an inductive subclass of itself.
- Any class $\{a_1, a_2, \dots, a_n\}$ (i.e., any string class defined by finite enumeration) is an inductive subclass of $\mathcal{A}^\circ$.

Some additional remarks

Let $\mathcal{A}^\circ$ be the class of all strings over an alphabet $\mathcal{A}$.

- The empty class $\emptyset$ is an inductive subclass of $\mathcal{A}^\circ$.
- $\mathcal{A}^\circ$ is an inductive subclass of itself.
- Any class $\{a_1, a_2, \dots, a_n\}$ (i.e., any string class defined by finite enumeration) is an inductive subclass of $\mathcal{A}^\circ$.
- Inductive classes are not closed for difference. Even the complement $\bar{B} = \mathcal{A}^\circ - B$ of an inductive class B is not necessarily inductive.

Some additional remarks

Let $\mathcal{A}^\circ$ be the class of all strings over an alphabet $\mathcal{A}$.

- The empty class $\emptyset$ is an inductive subclass of $\mathcal{A}^\circ$.
- $\mathcal{A}^\circ$ is an inductive subclass of itself.
- Any class $\{a_1, a_2, \dots, a_n\}$ (i.e., any string class defined by finite enumeration) is an inductive subclass of $\mathcal{A}^\circ$.
- Inductive classes are not closed for difference. Even the complement $\bar{B} = \mathcal{A}^\circ - B$ of an inductive class B is not necessarily inductive.

The first three remarks are trivial. The fourth one is extremely important for metalogic and will be proved (by examples) later.

Smullyan's mysterious automaton

Smullyan's mysterious automaton

We have a machine that works like a canonical calculus. (But we don't know the rules, we only see the outputs.) We know only that it prints strings of an alphabet $\mathcal{S} = \{\neg, P, N, (,)\}$.

Smullyan's mysterious automaton

We have a machine that works like a canonical calculus. (But we don't know the rules, we only see the outputs.) We know only that it prints strings of an alphabet $\mathcal{S} = \{\neg, P, N, (,)\}$.

We equip the strings with the following grammar and semantics:

Smullyan's mysterious automaton

We have a machine that works like a canonical calculus. (But we don't know the rules, we only see the outputs.) We know only that it prints strings of an alphabet $\mathcal{S} = \{\neg, P, N, (,)\}$.

We equip the strings with the following grammar and semantics:

- Sentences are strings of the form $P(X)$, $\neg P(X)$, $PN(X)$, $\neg PN(X)$, where X is any member of $\mathcal{S}^\circ$.

Smullyan's mysterious automaton

We have a machine that works like a canonical calculus. (But we don't know the rules, we only see the outputs.) We know only that it prints strings of an alphabet $\mathcal{S} = \{\neg, P, N, (,)\}$.

We equip the strings with the following grammar and semantics:

- Sentences are strings of the form $P(X)$, $\neg P(X)$, $PN(X)$, $\neg PN(X)$, where X is any member of $\mathcal{S}^\circ$.
- Norm of the string X is the string $\ulcorner X(X) \urcorner$.

Smullyan's mysterious automaton

We have a machine that works like a canonical calculus. (But we don't know the rules, we only see the outputs.) We know only that it prints strings of an alphabet $\mathcal{S} = \{\neg, P, N, (,)\}$.

We equip the strings with the following grammar and semantics:

- Sentences are strings of the form $P(X)$, $\neg P(X)$, $PN(X)$, $\neg PN(X)$, where X is any member of $\mathcal{S}^\circ$.
- Norm of the string X is the string $\ulcorner X(X) \urcorner$.
- The sentence $P(X)$ is true iff the string X gets (sometimes) printed by our machine; $PN(X)$ is true iff the norm of X , i.e. $X(X)$ will be printed sometimes.

Smullyan's mysterious automaton

We have a machine that works like a canonical calculus. (But we don't know the rules, we only see the outputs.) We know only that it prints strings of an alphabet $\mathcal{S} = \{\neg, P, N, (,)\}$.

We equip the strings with the following grammar and semantics:

- Sentences are strings of the form $P(X)$, $\neg P(X)$, $PN(X)$, $\neg PN(X)$, where X is any member of $\mathcal{S}^\circ$.
- Norm of the string X is the string $\ulcorner X(X) \urcorner$.
- The sentence $P(X)$ is true iff the string X gets (sometimes) printed by our machine; $PN(X)$ is true iff the norm of X , i.e. $X(X)$ will be printed sometimes.
- ' $\neg$ ' means negation.

Homework about Smullyan's machine

Prove that it is not possible for the machine to print all true sentences, but only those. (It may print strings that are not sentences, but we are only talking about sentences that the machine can print.) I.e., if it only prints true sentences, then there is at least one true sentence that it never prints.

Bonus: If you proved this proposition, you may propose a name for it.

The language of propositional logic

The language of propositional logic

We have an infinite sequence of propositional constants $p_0, p_1, \dots, p_n, \dots$ and two logical connectives: $\neg, \supset$. How to define the class of wff's as an inductively defined class over a finite alphabet, possibly avoiding the use of natural numbers?

The language of propositional logic

We have an infinite sequence of propositional constants $p_0, p_1, \dots, p_n, \dots$ and two logical connectives: $\neg, \supset$. How to define the class of wff's as an inductively defined class over a finite alphabet, possibly avoiding the use of natural numbers? Instead of p_n we write $\pi\iota\dots\iota$. We don't need numbers for indexes, but we do need to be able to distinguish between propositional constants.

The language of propositional logic

We have an infinite sequence of propositional constants $p_0, p_1, \dots, p_n, \dots$ and two logical connectives: $\neg, \supset$. How to define the class of wff's as an inductively defined class over a finite alphabet, possibly avoiding the use of natural numbers?

Instead of p_n we write $\pi\iota\dots\iota$. We don't need numbers for indexes, but we do need to be able to distinguish between propositional constants.

Alphabet: $\mathcal{A}_{PL} = \{ (,), \pi, \iota, \neg, \supset \}$. Auxiliary letters: I for *index* and F for *formula*. The calculus $K_{Language(PL)}$:

The language of propositional logic

We have an infinite sequence of propositional constants $p_0, p_1, \dots, p_n, \dots$ and two logical connectives: $\neg, \supset$. How to define the class of wff's as an inductively defined class over a finite alphabet, possibly avoiding the use of natural numbers?

Instead of p_n we write $\pi\iota\dots\iota$. We don't need numbers for indexes, but we do need to be able to distinguish between propositional constants.

Alphabet: $\mathcal{A}_{PL} = \{ (,), \pi, \iota, \neg, \supset \}$. Auxiliary letters: I for *index* and F for *formula*. The calculus $K_{Language(PL)}$:

1. $I\emptyset$
2. $Ix \rightarrow Ix\iota$
3. $Ix \rightarrow F\pi x$
4. $Fx \rightarrow F\neg x$
5. $Fx \rightarrow Fy \rightarrow F(x \supset y)$
- 5*. $Fx \rightarrow x$

Comments to the above calculus

Comments to the above calculus

- The numbers in the left column are for reference only; they are not part of the calculus.

Comments to the above calculus

- The numbers in the left column are for reference only; they are not part of the calculus.
- The sign of the empty word can be omitted from rule 1.

Comments to the above calculus

- The numbers in the left column are for reference only; they are not part of the calculus.
- The sign of the empty word can be omitted from rule 1.
- 5^* . is a *release rule*: it erases an auxiliary letter. We can define the wff's of PL as the $\mathcal{A}_{PL}^\circ$ – *strings* derivable in this calculus.

Comments to the above calculus

- The numbers in the left column are for reference only; they are not part of the calculus.
- The sign of the empty word can be omitted from rule 1.
- 5^* . is a *release rule*: it erases an auxiliary letter. We can define the wff's of PL as the $\mathcal{A}_{PL}^\circ$ – *strings* derivable in this calculus.
- The language of propositional logic could have been defined without using auxiliary letters (see textbook p. 40). But it is not always possible to eliminate the auxiliary letters and they make our work simpler and more transparent even if they (or some of them) are not necessary.

Propositional logic as calculus (informally)

Propositional logic as calculus (informally)

We want to have a calculus that derives the provable formulas of this language.

Propositional logic as calculus (informally)

We want to have a calculus that derives the provable formulas of this language.

We include the (first five rules of) previous calculus.

Propositional logic as calculus (informally)

We want to have a calculus that derives the provable formulas of this language.

We include the (first five rules of) previous calculus.

New auxiliary letter: L with the intended meaning „provable formula”.

Propositional logic as calculus (informally)

We want to have a calculus that derives the provable formulas of this language.

We include the (first five rules of) previous calculus.

New auxiliary letter: L with the intended meaning „provable formula”.

The rules after the definition of the formulas will be just the usual axiom schemes of propositional logic.

Propositional logic as calculus (informally)

We want to have a calculus that derives the provable formulas of this language.

We include the (first five rules of) previous calculus.

New auxiliary letter: L with the intended meaning „provable formula”.

The rules after the definition of the formulas will be just the usual axiom schemes of propositional logic.

The rule of detachment (for ‘ $\supset$ ’) will appear again as a rule of our calculus.

Propositional logic as calculus (informally)

We want to have a calculus that derives the provable formulas of this language.

We include the (first five rules of) previous calculus.

New auxiliary letter: L with the intended meaning „provable formula”.

The rules after the definition of the formulas will be just the usual axiom schemes of propositional logic.

The rule of detachment (for ‘ $\supset$ ’) will appear again as a rule of our calculus.

We need now a release rule for L .

The calculus of propositional logic K_{PL}

The calculus of propositional logic K_{PL}

The calculus begins with the first five rules of $K_{Language(PL)}$ and continues as follows:

The calculus of propositional logic K_{PL}

The calculus begins with the first five rules of $K_{Language(PL)}$ and continues as follows:

$$6. \quad Fu \rightarrow Fv \rightarrow L(u \supset (v \supset u))$$

$$7. \quad Fu \rightarrow Fv \rightarrow Fw \rightarrow L((u \supset (v \supset w)) \supset ((u \supset v) \supset (u \supset w)))$$

$$8. \quad Fu \rightarrow Fv \rightarrow L((\neg u \supset \neg v) \supset (v \supset u))$$

$$9. \quad Lu \rightarrow L(u \supset v) \rightarrow Lv$$

$$9^*. \quad Lx \rightarrow x$$

The calculus of propositional logic K_{PL}

The calculus begins with the first five rules of $K_{Language(PL)}$ and continues as follows:

- 6. $Fu \rightarrow Fv \rightarrow L(u \supset (v \supset u))$
- 7. $Fu \rightarrow Fv \rightarrow Fw \rightarrow L((u \supset (v \supset w)) \supset ((u \supset v) \supset (u \supset w)))$
- 8. $Fu \rightarrow Fv \rightarrow L((\neg u \supset \neg v) \supset (v \supset u))$
- 9. $Lu \rightarrow L(u \supset v) \rightarrow Lv$
- 9*. $Lx \rightarrow x$

This calculus defines the class of provable formulas of propositional logic (shortly: the propositional logic) L_{PL} .