Conceptual introduction to spacetime geometry

Differential geometry

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Smooth manifolds

Smooth manifold

Let M be a set.

- A m-dimensional chart on M is a pair (U, u) with
  - $U \subseteq M$
  - $u : U \to \mathbb{R}^m$ injective and $u(U) \subseteq \mathbb{R}^m$ open
- An m-dimensional atlas on M is a set of m-dimensional charts $\mathcal{A} = \{(U_\alpha, u_\alpha)\}$ such that
  - the charts cover M, that is $\bigcup_\alpha U_\alpha = M$
  - each pair of charts is "compatible," that is if $U_\alpha \cap U_\beta \neq \emptyset$ then $u_\beta \circ u_\alpha^{-1} : u_\alpha(U_\alpha \cap U_\beta) \to u_\beta(U_\alpha \cap U_\beta)$ is smooth[1]
- An atlas on M is maximal if it is not extendible by a chart that is compatible with each chart in the atlas.
- An m-dimensional smooth/differentiable manifold is pair $(M, \mathcal{A})$ where $\mathcal{A}$ is an m-dimensional maximal atlas on M.

[1] All its mixed partial derivatives to all orders exist and are continuous.

Motivating ideas

- A maximal atlas provides a "smooth/differentiable structure" to M (that is the notion to discern which functions to and from M are smooth/differentiable).
- $(M, \mathcal{A})$ is an "internal" characterization (one that isn't dependent on M being embedded in an ambient Euclidean space) of smooth surfaces.
- An m-dimensional atlas says that M "locally looks like $\mathbb{R}^m$."

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Smooth maps

Let M be an m-dimensional smooth manifold. We say that a map $f : M \to \mathbb{R}$ is smooth/differentiable at a point $x \in M$ if there is a chart (U, u) such that

- $x \in U$
- $f \circ u^{-1} : \mathbb{R}^m \rightarrowtail \mathbb{R}$ is smooth[2]

[2] All its mixed partial derivatives to all orders exist and are continuous.

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- $x \in U$
- $f \circ u^{-1} : \mathbb{R}^n \rightarrowtail \mathbb{R}$ is smooth

Note that this definition doesn't depend on the choice of chart: if (V, v) is another chart such that $x \in V$, then, since any two charts are compatible,

$$f \circ v^{-1} = \left(f \circ u^{-1}\right) \circ \left(u \circ v^{-1}\right)$$

is guaranteed to be smooth too.[3]

[3]The composition of smooth functions are smooth.

Smooth maps

Let M be an m-dimensional smooth manifold. We say that a map $f : M \to \mathbb{R}$ is smooth/differentiable if it is smooth at every point of M. This is equivalent with the fact that $f \circ u^{-1} : \mathbb{R}^m \rightarrowtail \mathbb{R}$ is smooth for every chart (U, u).

The set of smooth $M \to \mathbb{R}$ maps will be denoted by $C^\infty(M)$.

Smooth maps

Let M be an m-dimensional, N be an n-dimensional smooth manifold. We say that a map $f : M \to N$ is smooth/differentiable if

- for any chart (U, u) on M
- and any chart (V, v) on N
- with $f(U) \cap V \neq \emptyset$,

$v \circ f \circ u^{-1} : \mathbb{R}^m \rightarrowtail \mathbb{R}^n$ is smooth.

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We say that $f : M \to N$ is a diffeomorphism if it is a smooth bijection and its inverse is smooth too.

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Smooth curve in $\mathbb{R}^n$

$C \subseteq \mathbb{R}^n$ is a simple curve[4] if there is

- an open interval $I \subseteq \mathbb{R}$, and
- $p : I \to \mathbb{R}^n$ injective, smooth function with continuous inverse (the "parametrization" of C), such that
- $p(I) = C$

C is a curve if every point x of it has a neighborhood $B_x \subseteq \mathbb{R}^n$ such that $B_x \cap C$ is a simple curve.

[4] This basically means that a simple curve is something that can be obtained from a straight line segment by smoothly distorting it, and without creating loops.

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Facts

- The parametrization of a given curve is far from unique: take any open interval $J \subseteq \mathbb{R}$ and smooth bijection $\phi : J \to I$ with smooth inverse, then $p \circ \phi : J \to \mathbb{R}^n$ is also a parametrization.
- Assume that $p : I \to \mathbb{R}^n$ and $q : J \to \mathbb{R}^n$ are parametrizations of curve C, such that $\dot{p}(s) \neq \mathbf{0}$ and $\dot{q}(t) \neq \mathbf{0}$ for all $s \in I$ and $t \in J$. Then $p^{-1} \circ q : J \to I$ and $q^{-1} \circ p : I \to J$ are smooth.

Smooth submanifold in $\mathbb{R}^n$

$M \subseteq \mathbb{R}^n$ is an m-dimensional simple submanifold if there is

- an open set $D \subseteq \mathbb{R}^m$, and
- $p : D \to \mathbb{R}^n$ injective, smooth function with continuous inverse (the "parametrization" of M), such that
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Terminology

For a non-simple submanifold, a parametrization is also called local parametrization. The inverse of a local parametrization is a local coordinatization.

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Charts depict the Earth as subsets of the Euclidean plane

It is (roughly) a 2-dimensional sphere, but we use local charts to depict it as subsets of 2-dimensional Euclidean spaces. Note that such a chart will always give a somewhat distorted picture of the planet; the distances on the sphere are never quite correct, and either the areas or the angles (or both) are wrong. For example, in the standard maps of the world, Greenland always appears much bigger than it really is. (Do you know how its area compares to that of India?)

To describe the entire planet, one uses an atlas with a collection of such charts, such that every point on the planet is depicted in at least one such chart.

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This is a good picture to have in mind. It must be noted though that it doesn't quite correspond with the precise definition of a smooth manifold. The reason is that M without the atlas is just a bare set that has no notion of

- a "local neighborhood of its point"
- "distance," "area," or "angle" that could be faithfully or distortedly depicted by a chart.[5]

[5] Given an atlas on M, a $U \subseteq M$ can be regarded as a local neighborhood of a point $x \in M$, if $x \in U$ and (U, u) is a chart

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To describe the entire planet, one uses an atlas with a collection of such charts, such that every point on the planet is depicted in at least one such chart.

A better notion of "M locally looking like $\mathbb{R}^m$" is given by the so-called tangent space; which is what we now turn to.

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in the atlas. Distance, area, or angle are notions that don't even make sense even in the presence of an atlas in general. For this, we will need an extra structure on the manifold.

Tangent spaces

Elements of $\mathbb{R}^n$ in two roles

When we think of the elements of $\mathbb{R}^n$, we think of them either as

- points in space, whose only property is location, expressed by the coordinates $(x_1, ..., x_n)$, or
- vectors, which are objects that have magnitude and direction, but whose location is irrelevant.

One of the motivations to introduce the concept of an affine space is distinguishing precisely these two roles. In an affine space $(A, V, +)$, the points of the space are elements of A, and the vectors that have magnitude and direction, but no location, are elements of V. At each point of A, we have a separate copy of V.

Tangent vectors of a submanifold in $\mathbb{R}^n$

We considered the unit circle as a curve (one-dimensional submanifold) in $\mathbb{R}^2$. To emphasize the difference between points and vectors, we can consider an affine space $(A, \mathbb{R}^2, +)$ and a unit circle as a subset of A, with center at $x \in A$:[6]

$$S^1_x = \{y \in A \mid |y - x| = 1\}$$

At each $y \in S^1_x$, the unit circle has a line tangent to it. This is a one-dimensional linear subspace of $\mathbb{R}^2$ consisting of those vectors that are tangent to the circle at y, in other words, that are orthogonal to the radial unit vector through y. So the tangent line of S^1_x at an $y \in S^1_x$ is

$$T_y S^1_x = \left\{\mathbf{v} \in \mathbb{R}^2 \mid \pi\left(\mathbf{v}, y - x\right) = 0\right\}$$

where π denotes the scalar product in $\mathbb{R}^2$. One can think of $T_y S^1_x$ as a subspace of the copy of $\mathbb{R}^2$ at $y \in A$.

[6]Here $|y - x|$ denotes the length of the vector in $\mathbb{R}^2$ that points from point x to y.

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Notice that this definition of the tangent space is based on the fact there is an ambient affine space around our manifold whose vector space component $\mathbb{R}^2$ has a Euclidean structure. The following characterization of tangent vectors will lend itself to generalization for an arbitrary manifold without ambient space.

Vectors as derivations

Main idea: a vector can be identified with the directional derivative along that vector, as an operator on smooth functions.

Vectors as derivations

Let $(A, \mathbb{R}^n, +)$ be an affine space, and $f : A \to \mathbb{R}$ be a smooth function.[7] The directional derivative of f at an $x \in A$ along a $\mathbf{v} \in \mathbb{R}^n$ is

$$D_{\mathbf{v}} f(x) = \left.\frac{df(x + t\mathbf{v})}{dt}\right|_{t=0} = \lim_{t \to 0} \frac{f(x + t\mathbf{v}) - f(x)}{t}$$

One can consider the directional derivative as a map (operator) on functions:

$$D_{\mathbf{v}}^x : C^\infty(A) \to \mathbb{R}, \quad f \mapsto D_{\mathbf{v}} f(x)$$

where $C^\infty(A)$ is the space of all $A \to \mathbb{R}$ smooth functions.

The idea is that we can identify a vector $\mathbf{v}$ at a point x with this $D_{\mathbf{v}}^x$ operator.

[7] A function $f : A \to \mathbb{R}$ is smooth if all its mixed partial derivatives to all orders exist and are continuous. For a function whose domain is an affine space, its partial derivatives can be understood this way: $\partial_i f(x) = \lim_{t \to 0} \frac{f(x + t\mathbf{e}_i) - f(x)}{t}$, where $\mathbf{e}_i$ is the ith standard basis vector in $\mathbb{R}^n$.

Vectors as derivations

To make this idea precise, we first observe that $D^x_{\mathbf{v}}$ as a $C^\infty(A) \to \mathbb{R}$ map has the following two properties:[8]

- $D^x_{\mathbf{v}}(\alpha f + \beta g) = \alpha D^x_{\mathbf{v}}(f) + \beta D^x_{\mathbf{v}}(g)$ (linear)
- $D^x_{\mathbf{v}}(fg) = f(x) D^x_{\mathbf{v}}(g) + g(x) D^x_{\mathbf{v}}(f)$ (product rule)

[8] If $f, g \in C^\infty(A)$ then $\alpha f + \beta g, fg \in C^\infty(A)$.

Vectors as derivations

This motivates the following definition. Call any $D^x : C^\infty(A) \to \mathbb{R}$ map a derivation at x, if it has those two properties:

- $D^x(\alpha f + \beta g) = \alpha D^x(f) + \beta D^x(g)$ (linear)
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- $D^x(fg) = f(x) D^x(g) + g(x) D^x(f) \quad \text{(product rule)}$

We can now identify a vector as a derivation, based on the following:

Fact
The assignment
$$\mathbb{R}^n \to \{\text{derivations at } x\}, \quad \mathbf{v} \mapsto D^x_{\mathbf{v}}$$
is a bijection.

Tangent vectors in a smooth manifold

Let M be an m-dimensional smooth manifold, and $x \in M$. Consider $C^\infty(M)$, the space of all $M \to \mathbb{R}$ smooth functions.[9] We call a $D^x : C^\infty(M) \to \mathbb{R}$ map a derivation at x, if it has the following two properties:

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This definition makes sense, as many important features of differentiation can be proved merely on the basis of these two properties. For example:

- If $f : M \to \mathbb{R}$ is a constant function, then $D^x(f) = 0$ for any derivation D^x at any $x \in M$.
- A derivation is only sensitive to the local behavior of functions, in the sense that if $f, g \in C^{\infty}(M)$ are identical in a local neighborhood of $x \in M$,[10] then $D^x(f) = D^x(g)$.

[10] This means that there is chart (U, u) with $x \in U$, such that $f(y) = g(y)$ for all $y \in U$.

Proof of the previous two facts

- Let $f : M \to \mathbb{R}, f(x) = \alpha\,(\neq 0)$ for all $x \in M$. Due to linearity, $D^x(\alpha f) = \alpha D^x(f)$. By the product rule, $D^x(ff) = f(x)\,D^x(f) + f(x)\,D^x(f) = 2\alpha D^x(f)$. Since $\alpha f = ff$, we have $\alpha D^x(f) = 2\alpha D^x(f)$ which implies $D^x(f) = 0$.
- Let $x \in M$, and $f, g \in C^\infty(M)$ such that $f(y) = g(y)$ for all $y \in U$, where U is the domain of a chart around x. Take a $b \in C^\infty(M)$ such that $b(y) = 0$ for $y \notin U$, and $b(x) \neq 0$ (there exists such a $b \in C^\infty(M)$). By the product rule, we have

$$
\begin{aligned}
D^x(bf) &= b(x)\,D^x(f) + f(x)\,D^x(b) \\
D^x(bg) &= b(x)\,D^x(g) + g(x)\,D^x(b) = b(x)\,D^x(g) + f(x)\,D^x(b)
\end{aligned}
$$

  Observe that $bf = bg$, therefore $D^x(bf) = D^x(bg)$, which implies $D^x(f) = D^x(g)$.

Tangent vectors in a smooth manifold

Let M be an m-dimensional smooth manifold, and $x \in M$. Consider $C^\infty(M)$, the space of all $M \to \mathbb{R}$ smooth functions. We call a $D^x : C^\infty(M) \to \mathbb{R}$ map a derivation at x, if it has the following two properties:

- $D^x(\alpha f + \beta g) = \alpha D^x(f) + \beta D^x(g)$ (linear)
- $D^x(fg) = f(x) D^x(g) + g(x) D^x(f)$ (product rule)

Motivated by the identification of vectors and derivations in an affine space, we call a derivation at x a tangent vector of M at x. The set of all derivations at x is the tangent space of M at x, which will be denoted by $T_x M$.

Examples of tangent vectors

- Let $I \subseteq \mathbb{R}$ be open interval. We say that $\gamma : I \to M$ is a smooth curve on M if $u \circ \gamma : I \to \mathbb{R}^m$ is smooth for every chart (U, u) with $U \cap \gamma(I) \neq \emptyset$. Suppose for simplicity that $0 \in I$ and $\gamma(0) = x$. The tangent vector of γ at x is[11]

$$\dot{\gamma}_x : C^\infty(M) \to \mathbb{R}, \quad f \mapsto (f \circ \gamma)^{\cdot}(0)$$

- Let (U, u) be a chart on M, and $x \in U$. The derivation wrt. the kth coordinate in coordinates u at x is[12]

$$\left.\frac{\partial}{\partial u_k}\right|_x : C^\infty(M) \to \mathbb{R}, \quad f \mapsto \partial_k\left(f \circ u^{-1}\right)(u(x))$$

[11] Note that in an affine space, the directional derivative along a vector $\mathbf{v}$ is a special case of $\dot{\gamma}_x$ when the curve is a straight line pointing in the $\mathbf{v}$ direction: $\gamma(t) = x + t\mathbf{v}$.

[12] Note that $\left.\frac{\partial}{\partial u_k}\right|_x$ is the tangent vector of the curve $\gamma : (\varepsilon, \varepsilon) \to M, \quad t \mapsto u^{-1}(u(x) + t\mathbf{e}_k)$ (the pre-image of the kth coordinate curve through $u(x)$ in coordinates u) in the previous sense, where ε is a small number and $\mathbf{e}_k$ is the kth standard basis vector in $\mathbb{R}^m$.

Tangent space as a vector space

T_xM is a vector space under the operations

$$\left(D_1^x + D_2^x\right)(f) = D_1^x(f) + D_2^x(f)$$
$$\left(\alpha D^x\right)(f) = \alpha \cdot D^x(f)$$

Moreover, if M is an m-dimensional manifold, then the dimension of T_xM is m. In particular, if (U, u) is a chart around x, then

$$\left.\frac{\partial}{\partial u_1}\right|_x, \ldots, \left.\frac{\partial}{\partial u_m}\right|_x$$

forms a basis of T_xM.

Sketch of proof that the $\frac{\partial}{\partial u_i}\Big|_x$-s form a basis

To see that they are linearly independent, we have to show that $\sum_{i=1}^{m} \alpha_i \frac{\partial}{\partial u_i}\Big|_x = \mathbf{0}$ implies $\alpha_k = 0$ for all k. $\sum_{i=1}^{m} \alpha_i \frac{\partial}{\partial u_i}\Big|_x = \mathbf{0}$ means that $\sum_{i=1}^{m} \alpha_i \frac{\partial}{\partial u_i}\Big|_x (f) = 0$ for all $f \in C^\infty(M)$. Let u_k denote $\mathrm{pr}_k \circ u$, where $\mathrm{pr}_k : \mathbb{R}^m \to \mathbb{R}, (x_1, ..., x_k, ...x_m) \mapsto x_k$, the projection onto the kth coordinate. We will plug u_k in the place of f.[13] But first observe that

$$\frac{\partial}{\partial u_i}\Bigg|_x (u_k) = \partial_i \left(u_k \circ u^{-1}\right)(u(x)) = \partial_i \left(\mathrm{pr}_k \circ u \circ u^{-1}\right)(u(x)) = \partial_i \mathrm{pr}_k (u(x)) = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases} \tag{1}$$

Now, applying this, we have $\sum_{i=1}^{m} \alpha_i \frac{\partial}{\partial u_i}\Big|_x (u_k) = \alpha_k$, which indeed means that $\alpha_k = 0$ must hold for all k, if $\sum_{i=1}^{m} \alpha_i \frac{\partial}{\partial u_i}\Big|_x (f) = 0$ for all $f \in C^\infty(M)$.

To see that any $D^x \in T_xM$ can be expressed as a linear combination of the $\frac{\partial}{\partial u_i}\Big|_x$-s, one can show that

$$D^x = \sum_{i=1}^{m} D^x(u_i) \frac{\partial}{\partial u_i}\Bigg|_x$$

[13]Note that strictly speaking $u_k \notin C^\infty(M)$ since the domain of u_k is not the whole M, but only U. But we have seen that derivations D^x are only sensitive to the local behavior of functions around x, so we can plug into D^x any function that is defined in a neighborhood of x. And u_k is such a function.

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$$\left.\frac{\partial}{\partial u_1}\right|_x, \ldots, \left.\frac{\partial}{\partial u_m}\right|_x$$

forms a basis of T_xM.

Note that in an affine space $(A, V, +)$, at every point x of A we have a copy of V. In a manifold M, at every point x of M we have the vector space T_xM, but for different points x and y, T_xM and T_yM have nothing to do to each other.

Vector fields

- $TM := \bigcup_{x \in M} T_x M$ is called the tangent bundle of M
- A vector field is a map $X : M \to TM$ such that for all $x \in M$, $X_x \in T_x M$
- A vector field is smooth if for all $f \in C^\infty(M)$ the map

  $$X(f) : M \to \mathbb{R}, \quad x \mapsto X_x(f)$$

  is smooth

Riemannian and Lorentzian manifolds

Riemannian manifold: motivating ideas

- A smooth manifold locally looks like its tangent space
- We want to say that a manifold locally looks like a Euclidean space (on a smooth surface, like the surface of the Earth, we can measure distances, areas, angles, etc.)
- We have seen that a Euclidean space can be described as a vector space that has a Euclidean scalar product on it

Recall: Euclidean geometry in m dimensions

$$(V, p)$$

where

- V is an m dimensional vector space
- p is a Euclidean scalar product on it

that is:

1. For all $\mathbf{u}, \mathbf{v} \in V$, $p(\mathbf{u}, \mathbf{v}) = p(\mathbf{v}, \mathbf{u})$
2. For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $p(\mathbf{u}, \mathbf{v} + \mathbf{w}) = p(\mathbf{u}, \mathbf{v}) + p(\mathbf{u}, \mathbf{w})$
3. For all $\alpha \in \mathbb{R}$ and all $\mathbf{u}, \mathbf{v} \in V$, $p(\mathbf{u}, \alpha\mathbf{v}) = \alpha p(\mathbf{u}, \mathbf{v})$
4. For all non-zero $\mathbf{u} \in V$, there is a $\mathbf{v} \in V$ such that $p(\mathbf{u}, \mathbf{v}) \neq 0$
5. The signature of p is $(m, 0)$ (i.e. all vectors of an orthonormal basis are of positive square length)

Riemannian manifold: motivating ideas

- A smooth manifold locally looks like its tangent space
- We want to say that a manifold locally looks like a Euclidean space (on a smooth surface, like the surface of the Earth, we can measure distances, areas, angles, etc.)
- We have seen that a Euclidean space can be described as a vector space that has a Euclidean scalar product on it
- So, we put a Euclidean scalar product on each tangent space of the manifold

Riemannian manifold

$$(M, g)$$

where

- M is an m dimensional smooth manifold
- g (called "metric" or "metric field") is a map that assigns a Euclidean scalar product to each tangent space of M

That is, if g_x denotes the Euclidean scalar product on T_xM, then

1. For all $\mathbf{x}, \mathbf{y} \in T_xM$, $g_x(\mathbf{x}, \mathbf{y}) = g_x(\mathbf{y}, \mathbf{x})$
2. For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in T_xM$, $g_x(\mathbf{x}, \mathbf{y} + \mathbf{z}) = g_x(\mathbf{x}, \mathbf{y}) + g_x(\mathbf{x}, \mathbf{z})$
3. For all $\alpha \in \mathbb{R}$ and all $\mathbf{x}, \mathbf{y} \in T_xM$, $g_x(\mathbf{x}, \alpha\mathbf{y}) = \alpha g_x(\mathbf{x}, \mathbf{y})$
4. For all non-zero $\mathbf{x} \in T_xM$, there is a $\mathbf{y} \in T_xM$ such that $g_x(\mathbf{x}, \mathbf{y}) \neq 0$
5. The signature of g_x is $(m, 0)$ (i.e. all vectors of an orthonormal basis are of positive square length)

Riemannian manifold

$$(M, g)$$

where

- M is an m dimensional smooth manifold
- g (called "metric" or "metric field") is a map that assigns a Euclidean scalar product to each tangent space of M

We require g be smooth, in the sense that for any smooth vector fields X and Y on M, the function

$$g(X, Y) : M \to \mathbb{R}, \quad x \mapsto g_x(X_x, Y_x)$$

be smooth.

Length

- The length of a vector $\mathbf{x} \in T_x M$ is

$$|\mathbf{x}| = \sqrt{g_x(\mathbf{x}, \mathbf{x})}$$

- The length of a smooth curve[14] $\gamma : [t_1, t_2] \to M$ is

$$|\gamma| = \int_{t_1}^{t_2} \sqrt{g_{\gamma(t)}\left(\dot{\gamma}_{\gamma(t)}, \dot{\gamma}_{\gamma(t)}\right)} dt$$

Length is invariant under reparametrization.[15]

---

[14] Officially, we have taken a smooth curve $\gamma : I \to M$ to be defined on an *open* interval I. We can extend this definition for a *closed* I: we say that $\gamma : [t_1, t_2] \to M$ is smooth if it has an extension to a smooth curve on an open domain, defined in a neighborhood of each endpoint t_1, t_2.

[15] One can think of the parametrization of a curve as a point moving on the curve through time, such that at time $t \in [t_1, t_2]$ it is at point $\gamma(t)$. Then $\dot{\gamma}_{\gamma(t)}$ is the velocity, $\sqrt{g_{\gamma(t)}\left(\dot{\gamma}_{\gamma(t)}, \dot{\gamma}_{\gamma(t)}\right)}$ is the speed of the point at time t. $\sqrt{g_{\gamma(t)}\left(\dot{\gamma}_{\gamma(t)}, \dot{\gamma}_{\gamma(t)}\right)} dt$ is the distance covered by the point during a small time interval dt around time t. The fact that length is invariant under reparametrization means that the length of the curve doesn't depend on how (fast) the point moves on it.

Length

- A curve $\gamma : [t_1, t_2] \to M$ is parametrized by arc length if for all $t \in [t_1, t_2]$, $\sqrt{g_{\gamma(t)}\left(\dot{\gamma}_{\gamma(t)}, \dot{\gamma}_{\gamma(t)}\right)} = 1$. In this case

$$|\gamma| = \int_{t_1}^{t_2} 1 dt = t_2 - t_1$$

Any smooth curve can be reparametrized by arc length.

Distance

- A curve from x to y is a curve $\gamma : [t_1, t_2] \to M$ with $\gamma(t_1) = x, \gamma(t_2) = y$. The distance between $x, y \in M$ is

$$d(x, y) = \text{greatest lower bound}\left\{|\gamma| \mid \gamma \text{ is a smooth curve from } x \text{ to } y\right\}$$

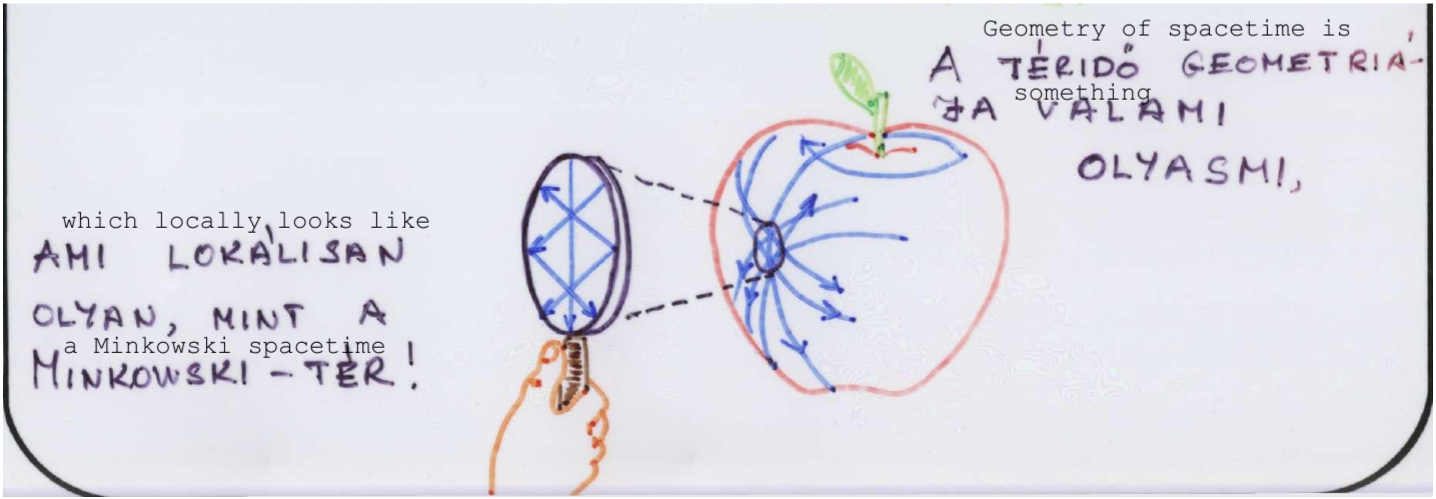
- A curve $\gamma : I \to M$ is minimal or "locally shortest" if there is an $\varepsilon > 0$ such that for all $\tau_1, \tau_2 \in I, |\tau_2 - \tau_1| < \varepsilon$, the length of the segment of γ between τ_1 and τ_2 is equal with $d(\gamma(\tau_1), \gamma(\tau_2))$—that is, in case the segment of γ between τ_1 and τ_2 is the "shortest path" between $\gamma(\tau_1)$ and $\gamma(\tau_2)$.

  Example: Consider two points on the "equator" of a two dimensional sphere that are not antipodal to one another. An equatorial curve running from one to the other the "long way" qualifies as a minimal curve. It is not the shortest curve between the two points (the shortest one is the equatorial curve running between them the "short way"), but any local modification of that curve will yield a longer curve.

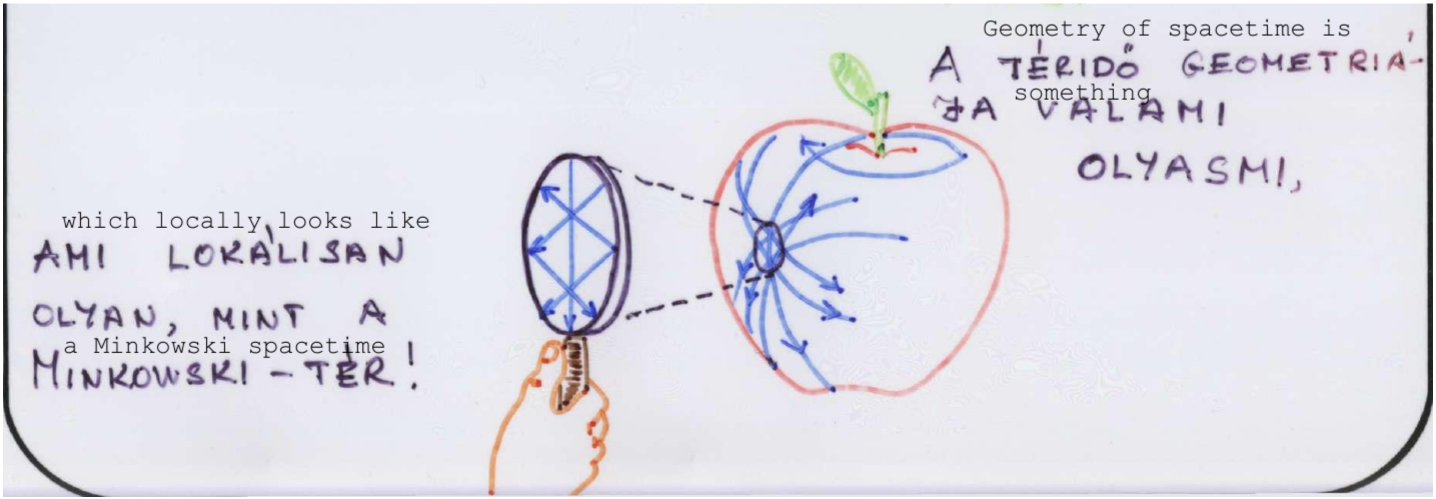
Lorentzian manifold: motivating ideas

- A smooth manifold locally looks like its tangent space
- We want to say that spacetime locally looks like Minkowski space
- We have seen that Minkowski space can be described as a vector space that has a Lorentz product on it

Spacetime is locally Minkowski-like



Spacetime is locally Minkowski-like



Note that what this means is in not completely obvious. Cf:
S.C. Fletcher and J. O. Weatherall: The local validity of special relativity, part 1: Geometry. *Philosophy of Physics*, 1(7), 2023.

Lorentzian manifold: motivating ideas

- A smooth manifold locally looks like its tangent space
- We want to say that spacetime locally looks like Minkowski space
- We have seen that Minkowski space can be described as a vector space that has a Lorentz product on it

Recall: Minkowski geometry

$$(V, \eta)$$

where

- V is a 4 dimensional vector space
- η is a Lorentz product on it

that is:

1. For all $\mathbf{u}, \mathbf{v} \in V$, $\eta(\mathbf{u}, \mathbf{v}) = \eta(\mathbf{v}, \mathbf{u})$
2. For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $\eta(\mathbf{u}, \mathbf{v} + \mathbf{w}) = \eta(\mathbf{u}, \mathbf{v}) + \eta(\mathbf{u}, \mathbf{w})$
3. For all $\alpha \in \mathbb{R}$ and all $\mathbf{u}, \mathbf{v} \in V$, $\eta(\mathbf{u}, \alpha\mathbf{v}) = \alpha\eta(\mathbf{u}, \mathbf{v})$
4. For all non-zero $\mathbf{u} \in V$, there is a $\mathbf{v} \in V$ such that $\eta(\mathbf{u}, \mathbf{v}) \neq 0$
5. The signature of η is $(1, 3)$

Lorentzian manifold: motivating ideas

- A smooth manifold locally looks like its tangent space
- We want to say that spacetime locally looks like Minkowski space
- We have seen that Minkowski space can be described as a vector space that has a Lorentz product on it
- So, we put a Lorentz product on each tangent space of the manifold

Lorentzian manifold

$$(M, g)$$

where

- M is an m dimensional smooth manifold
- g (called "metric" or "metric field") is a map that assigns a Lorentz product to each tangent space of M

That is, if g_x denotes the Lorentz product on T_xM, then

1. For all $\mathbf{x}, \mathbf{y} \in T_xM$, $g_x(\mathbf{x}, \mathbf{y}) = g_x(\mathbf{y}, \mathbf{x})$
2. For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in T_xM$, $g_x(\mathbf{x}, \mathbf{y} + \mathbf{z}) = g_x(\mathbf{x}, \mathbf{y}) + g_x(\mathbf{x}, \mathbf{z})$
3. For all $\alpha \in \mathbb{R}$ and all $\mathbf{x}, \mathbf{y} \in T_xM$, $g_x(\mathbf{x}, \alpha\mathbf{y}) = \alpha g_x(\mathbf{x}, \mathbf{y})$
4. For all non-zero $\mathbf{x} \in T_xM$, there is a $\mathbf{y} \in T_xM$ such that $g_x(\mathbf{x}, \mathbf{y}) \neq 0$
5. The signature of g_x is $(1, m-1)$

Lorentzian manifold

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We require g be smooth, in the sense that for any smooth vector fields X and Y on M, the function

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be smooth.

Lorentzian manifold

$$(M, g)$$

where

- M is an m dimensional smooth manifold
- g (called "metric" or "metric field") is a map that assigns a Lorentz product to each tangent space of M

Spacetime is a 4 dimensional Lorentzian manifold.

Causal structure

At any $x \in M$, g_x provides a light cone structure on $T_x M$. In particular, $\mathbf{x} \in T_x M$ is

- time-like iff $g_x(\mathbf{x}, \mathbf{x}) > 0$
- light-like iff $g_x(\mathbf{x}, \mathbf{x}) = 0$
- causal iff $g_x(\mathbf{x}, \mathbf{x}) \geq 0$
- space-like iff $g_x(\mathbf{x}, \mathbf{x}) < 0$

Causal structure

The classification extends naturally to curves. A smooth curve $\gamma : I \to M$ is

- time-like iff $g_{\gamma(t)}\left(\dot{\gamma}_{\gamma(t)}, \dot{\gamma}_{\gamma(t)}\right) > 0$, for all $t \in I$
- light-like iff $g_{\gamma(t)}\left(\dot{\gamma}_{\gamma(t)}, \dot{\gamma}_{\gamma(t)}\right) = 0$, for all $t \in I$
- causal iff $g_{\gamma(t)}\left(\dot{\gamma}_{\gamma(t)}, \dot{\gamma}_{\gamma(t)}\right) \geq 0$, for all $t \in I$
- space-like iff $g_{\gamma(t)}\left(\dot{\gamma}_{\gamma(t)}, \dot{\gamma}_{\gamma(t)}\right) < 0$, for all $t \in I$

These properties are preserved under reparametrization. So the classification also extends to images of smooth curves.

Interpretative principles

Recall that special relativity can be understood as Minkowski space endowed with the following interpretative principles:

- Time-like curves represent the spacetime trajectories of massive point particles, i.e., point particles with non-zero mass.
- Time-like lines represent the spacetime trajectories of free massive point particles, i.e., massive point particles that are not subject to any force.
- Light-like lines represent the spacetime trajectories of light rays.
- Minkowski distance between time-like separated events is time measured by a clock moving inertially between those events.
- Minkowski distance between space-like separated events is spatial distance measured with moving rods by an observer for whom those events are simultaneous.

Interpretative principles

Recall that special relativity can be understood as Minkowski space endowed with the following interpretative principles:

- Time-like curves (or more precisely, their images) represent the spacetime trajectories of massive point particles, i.e., point particles with non-zero mass.
- Time-like lines represent the spacetime trajectories of free massive point particles, i.e., massive point particles that are not subject to any force.
- Light-like lines represent the spacetime trajectories of light rays.
- Minkowski distance between time-like separated events is time measured by a clock moving inertially between those events.
- Minkowski distance between space-like separated events is spatial distance measured with moving rods by an observer for whom those events are simultaneous.

The first one is also a fundamental principle of general relativity, which we are now in position to formulate.

Affine connection

The problem of change and identity

A TÉRIDŐ KÜLÖNBÖZŐ PONTJAIBAN LÉVŐ

The things in different spacetime points are *ab ovo* different.

DOLGOK ELEVE KÜLÖNBÖZŐK.

At the same time, physics uses such phrases as

A FIZIKA TELE VAN OLYAN KIJELENTÉSEKKEL MINT:

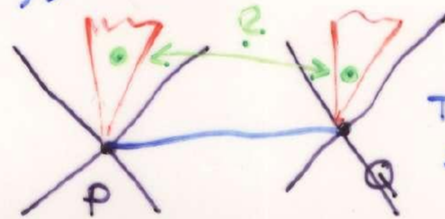
something is constant ', ' changes ', ' the gradient of ... '

"VALAMI ÁLLANDÓ", "VÁLTOZIK", "GRADIENSE = ..."

etc.

"$\partial_\mu A = \ldots$", "$L(\psi, \partial_\mu \psi,)$" STB.

?



space-like

TÉRSZERŰEN

separated

SZEPARÁLTAK

Tangent spaces at different points are (almost) disjoint

- The $C^\infty(M) \to \mathbb{R}, \quad f \mapsto 0$ map is an element of all tangent spaces, because it's trivially linear and satisfies the product rule for all $x \in M$. It is the $\mathbf{0}$ element of all tangent spaces as vector spaces.
- But there's no other shared element of the tangent spaces. That is, if

$$\mathbf{X} \in T_x M \cap T_y M$$

for $x \neq y$, then $\mathbf{X} = \mathbf{0}$.

Proof of $\mathbf{X} \in T_xM \cap T_yM$ for $x \neq y$ implies $\mathbf{X} = \mathbf{0}$

To see this, first take

- a neighborhood U around x, and V around y, such that $U \cap V = \emptyset$ (this is possible if the manifold is so-called Hausdorff, where every pair of distinct points have non-overlapping neighborhoods)
- functions $f, g \in C^\infty(M)$ such that f is zero outside U and g is zero outside V, and $f(x) \neq 0$ and $g(y) \neq 0$ (there exists such functions)

fg is zero everywhere, so $\mathbf{X}(fg) = 0$, due to the fact that to a constant function any derivation assigns 0. Now apply the product rule for $\mathbf{X}(fg)$ at point x:

$$0 = \mathbf{X}(fg) = f(x)\mathbf{X}(g) + g(x)\mathbf{X}(f) = f(x)\mathbf{X}(g) + 0 \cdot X(f) = f(x)\mathbf{X}(g)$$

Since $f(x) \neq 0$, $\mathbf{X}(g) = 0$.
Next take an arbitrary $h \in C^\infty(M)$, and apply the product rule for $\mathbf{X}(gh)$ at points x and y:

$$\begin{aligned}\mathbf{X}(gh) &= g(x)\mathbf{X}(h) + h(x)\mathbf{X}(g) = 0 \cdot \mathbf{X}(h) + h(x) \cdot 0 = 0 \\ &= g(y)\mathbf{X}(h) + h(y)\mathbf{X}(g) = g(y)\mathbf{X}(h) + h(y) \cdot 0 = g(y)\mathbf{X}(h)\end{aligned}$$

Since $g(y) \neq 0$, $\mathbf{X}(h) = 0$. h was arbitrarily chosen, so $\mathbf{X} = \mathbf{0}$.

Can we identify tangent spaces via charts?

Let (U, u) be a chart on M.

- For all $x \in U$,

  $$\left.\frac{\partial}{\partial u_1}\right|_x, ..., \left.\frac{\partial}{\partial u_m}\right|_x$$

  forms a basis of $T_x M$

- Relative this basis, each vector $\mathbf{X} \in T_x M$, for all $x \in U$, will have some coordinates $(X_1, ..., X_m) \in \mathbb{R}^m$, defined by the expansion of $\mathbf{X}$ in that basis:

  $$\mathbf{X} = \sum_{i=1}^{m} X_i \left.\frac{\partial}{\partial u_i}\right|_x$$

- One might want to identify an $\mathbf{X} \in T_x M$ and an $\mathbf{Y} \in T_y M$ $(x, y \in U, x \neq y)$ iff they have the same coordinates:

  $$(X_1, ..., X_m) = (Y_1, ..., Y_m)$$

Can we identify tangent spaces via charts?

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- One might want to identify an $\mathbf{X} \in T_x M$ and an $\mathbf{Y} \in T_y M$ $(x, y \in U, x \neq y)$ iff they have the same coordinates:
$$(X_1, \ldots, X_m) = (Y_1, \ldots, Y_m)$$

The problem with this identification is that it is chart-dependent.

Transformation of coordinates of vectors

Let (U, u) and (U', u') be charts such that $U \cap U' \neq \emptyset$. Then for any $x \in U \cap U'$, both $\left.\frac{\partial}{\partial u_1}\right|_x, \ldots, \left.\frac{\partial}{\partial u_m}\right|_x$ and $\left.\frac{\partial}{\partial u'_1}\right|_x, \ldots, \left.\frac{\partial}{\partial' u_m}\right|_x$ form a basis of $T_x M$. What is the transformation between the coordinates $X_1, \ldots, X_m$ and $X'_1, \ldots, X'_m$ of a vector $X \in T_x M$ in these two bases?

We have $X = \sum_{i=1}^{m} X_i \left.\frac{\partial}{\partial u_i}\right|_x = \sum_{j=1}^{m} X'_j \left.\frac{\partial}{\partial u'_j}\right|_x$ meaning that for all $f \in C^\infty(M)$

$$\mathbf{X}(f) = \sum_{i=1}^{m} X_i \left.\frac{\partial}{\partial u_i}\right|_x (f) = \sum_{j=1}^{m} X'_j \left.\frac{\partial}{\partial u'_j}\right|_x (f)$$

Plugging $f = u'_k = \mathrm{pr}_k \circ u'$, and using (1), yields

$$\sum_{i=1}^{m} X_i \left.\frac{\partial}{\partial u_i}\right|_x (u'_k) = \sum_{j=1}^{m} X'_j \left.\frac{\partial}{\partial u'_j}\right|_x (u'_k) = X'_k$$

That is, applying the definition of $\left.\frac{\partial}{\partial u_i}\right|_x$, the coordinate transformation is

$$X'_k = \sum_{i=1}^{m} X_i \partial_i \left(u'_k \circ u^{-1}\right)(u(x))$$

which one often writes as

$$X'_k = \sum_{i=1}^{m} X_i \frac{\partial u'_k}{\partial u_i}(u(x))$$

Transformation of coordinates of vectors

$$X'_k = \sum_{i=1}^{m} X_i \frac{\partial u'_k}{\partial u_i} (u(x))$$

or, equivalently,

$$\mathbf{X}' = \mathbf{J}(u(x))\,\mathbf{X}$$

where $\mathbf{J}(u(x))$ is the Jacobian matrix of $u' \circ u^{-1} : \mathbb{R}^m \rightarrowtail \mathbb{R}^m$ at $u(x)$.

The point is that this transformation depends on x, so

$$(X_1, ..., X_m) = (Y_1, ..., Y_m) \quad \not\Rightarrow \quad (X'_1, ..., X'_m) = (Y'_1, ..., Y'_m)$$

Transformation of coordinates of vectors