Conceptual introduction to spacetime geometry

From relativistic physical phenomena to Minkowski geometry

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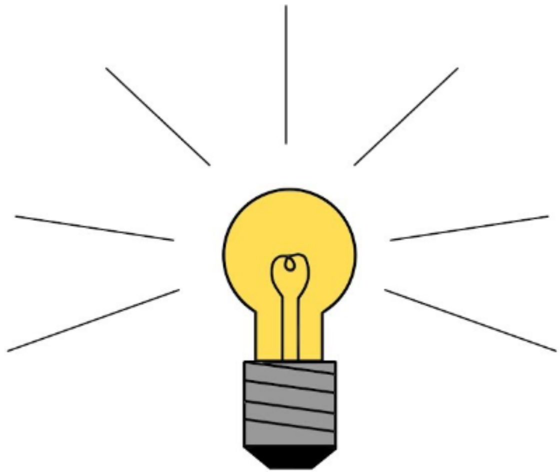
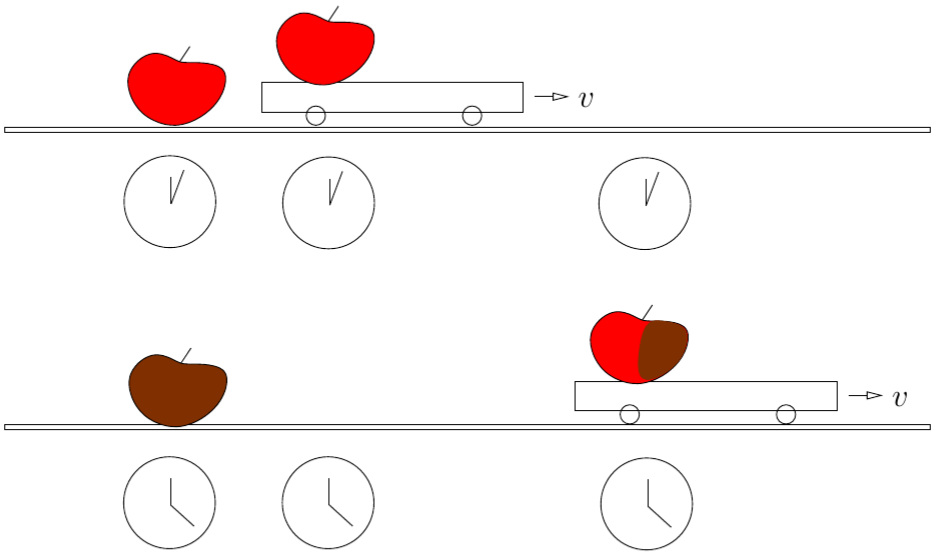
Relativistic phenomena

Speed of light is same in all directions and independent of its source: c



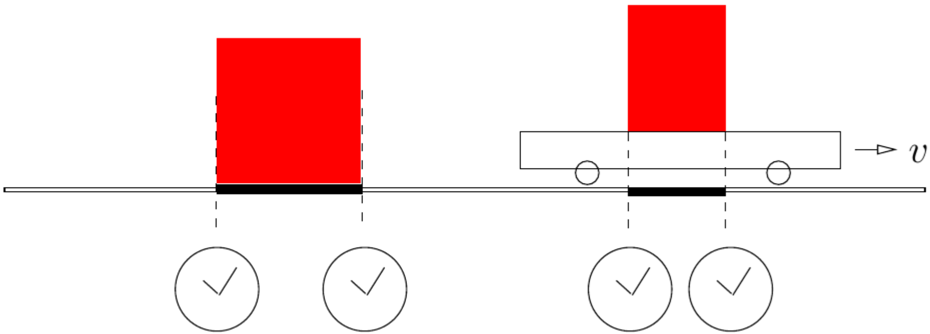
Processes in motion slow down: time dilation



$$T_{moving} = \frac{T_{rest}}{\sqrt{1-\frac{v^2}{c^2}}}$$

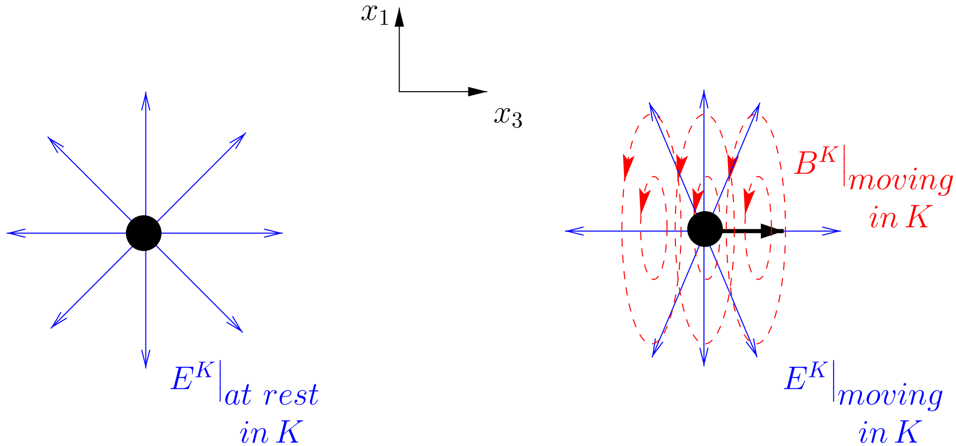
$c = 3 \cdot 10^8 \frac{m}{s}$
e.g. $v = 10 \frac{m}{s}$
$\frac{T_{moving} - T_{rest}}{T_{rest}} \approx 10^{-15} s$

Bodies in motion shrink: length contraction



$$L_{moving} = L_{rest}\sqrt{1 - \frac{v^2}{c^2}}$$

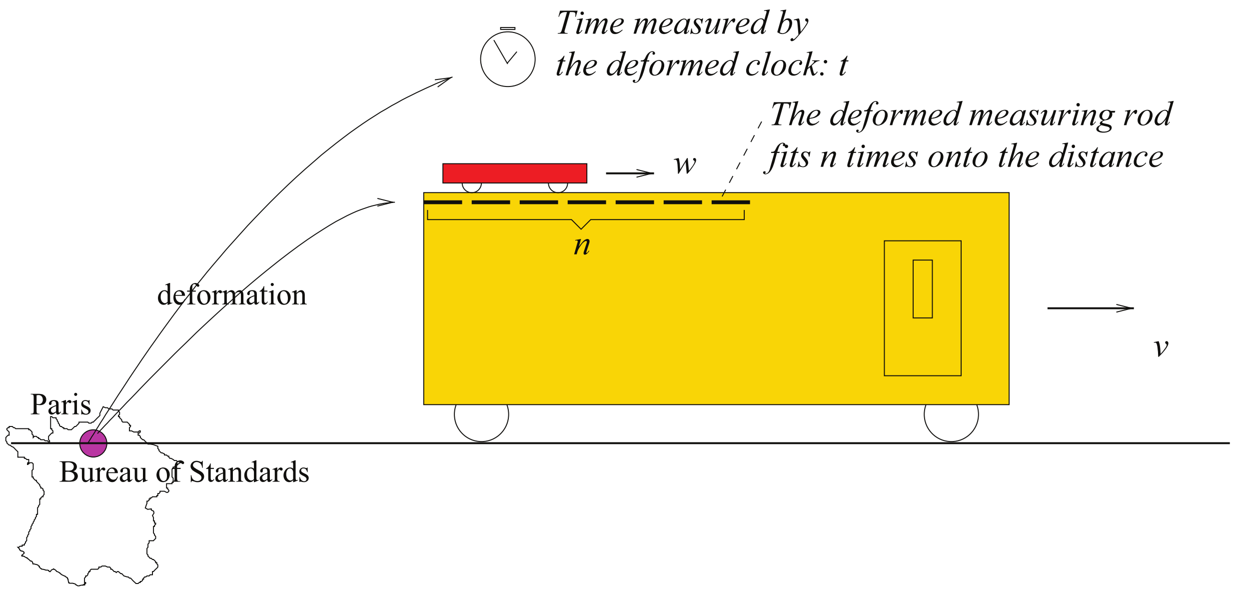
Contraction of the EM field of a point charge in motion



Electromagnetic field of a point charge at rest

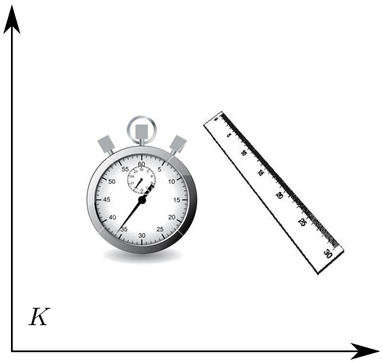
Electromagnetic field of a point charge moving in x_3-direction

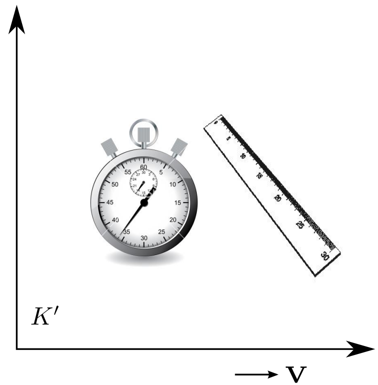
Clocks and measuring rods also suffer relativistic deformations



The connection of space and time coordinates of events as determined in a stationary vs. moving reference frame gets messed up. How exactly?

Standard 2D setup of two inertial frames K and K' in relative uniform motion





Time coordinate in K

A light signal is sent from the standard clock at clock reading t_1 to the locus of event A, such that the signal is reflected just at the moment of the occurrence of A. Upon receiving the reflected signal, let the clock reading be t_2. The time coordinate $t\,(A)$ is

$$t(A) := t_1 + \frac{1}{2}(t_2 - t_1)$$

Space coordinate in K

The space coordinate $x(A)$ of event A is the distance from the origin of K of the locus of A along the x-axis (straight line is usually defined by a light beam) measured by superposing the standard measuring rod, being always at rest relative to K.

Time coordinate in K'

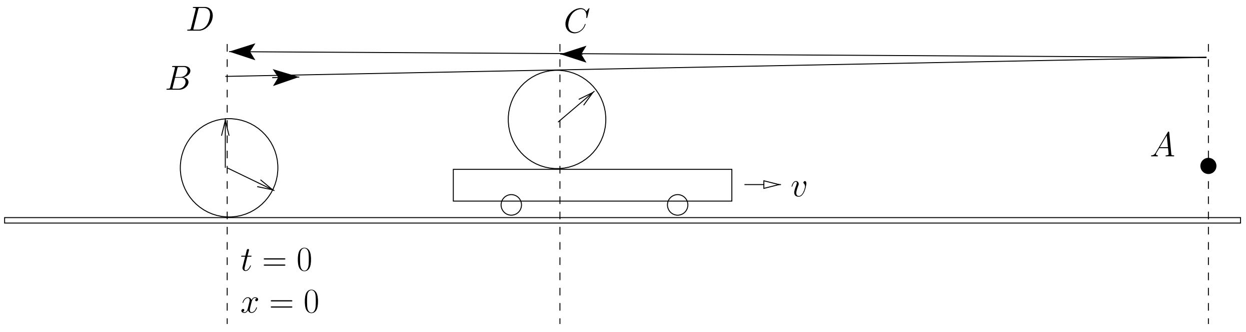
We take a copy of the standard clock at rest in K and accelerate it to frame K' (very gently so that it doesn't break!). Then we repeat the same operation as in above: A light signal is sent from the clock at clock reading t_1 to the locus of event A, such that the signal is reflected just at the moment of the occurrence of A. Upon receiving the reflected signal, let the clock reading be t_2. The time coordinate $t'(A)$ is

$$t'(A) := t_1 + \frac{1}{2}(t_2 - t_1)$$

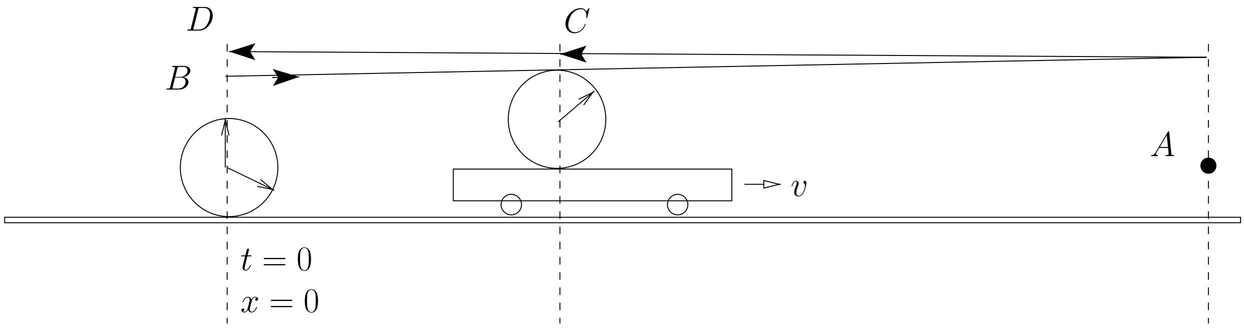
Space coordinate in K'

The space coordinate $x'(A)$ of event A is the distance from the origin of K' of the locus of A along the x-axis measured by superposing the standard measuring-rod, being always at rest relative to K', just the same way as if all were at rest.

We calculate $x'(A)$ and $t'(A)$ in terms of frame K



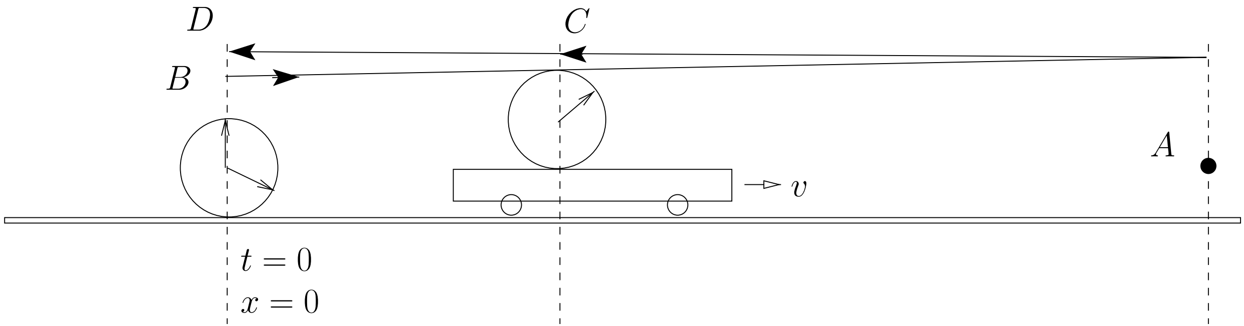
We calculate $x'(A)$ and $t'(A)$ in terms of frame K



We assume that the following hold in K:

1. Speed of light is the same in all directions and independent of its source
2. Time dilation and length contraction
3. Galilean kinematics (wrt. the space and time coordinates measured by rods and clocks at rest in K)

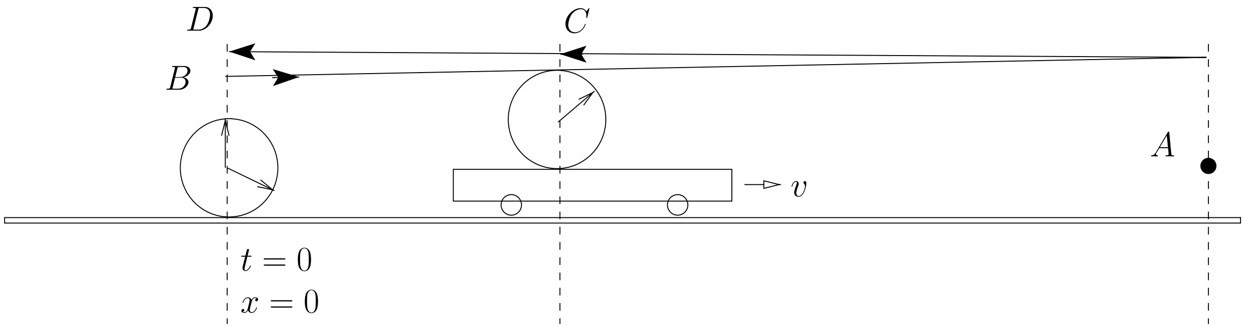
We calculate $x'(A)$ and $t'(A)$ in terms of frame K



By definition,

$$t(A) = \frac{t(D)}{2}$$

We calculate $x'(A)$ and $t'(A)$ in terms of frame K



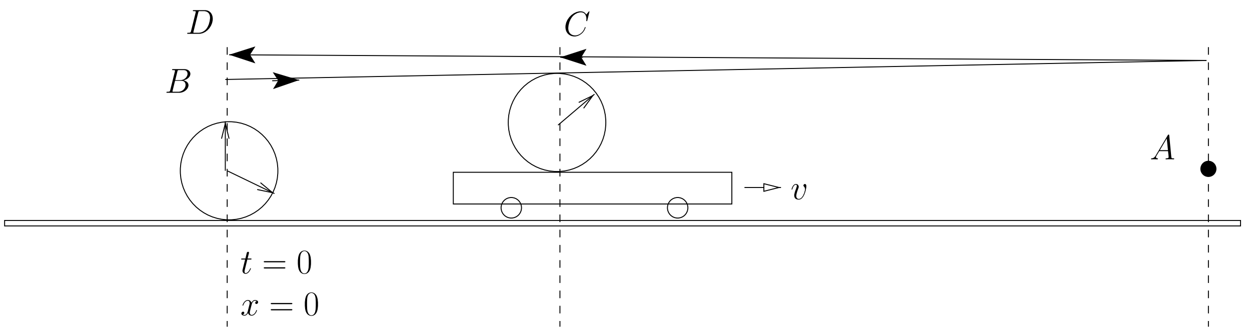
We know that

$$vt(C) = x(A) - c\,(t(C) - t(A))$$

and $x(A) = ct(A)$. Therefore,

$$t(C) = \frac{2ct(A)}{c+v}$$

We calculate $x'(A)$ and $t'(A)$ in terms of frame K



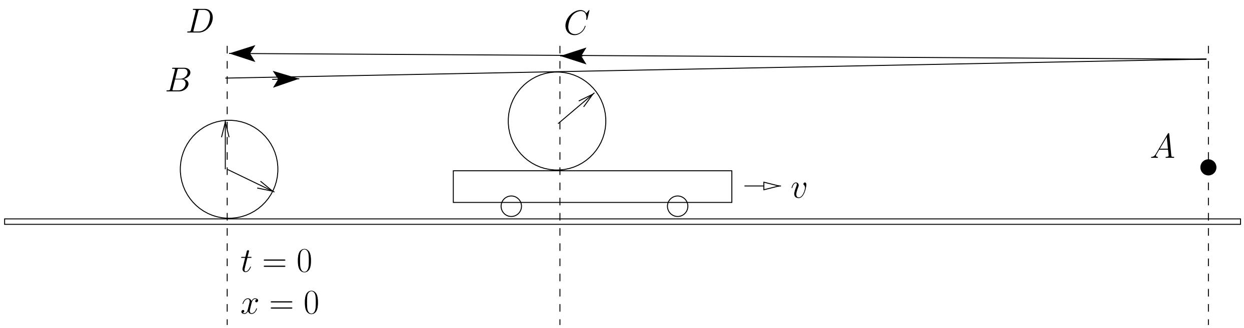
Taking into account that the moving observer's clock reading at C is

$$reading(C) \;=\; t(C)\sqrt{1-\frac{v^2}{c^2}}$$

we have

$$t'(A) \;=\; \frac{reading(C)}{2} = \frac{ct(A)}{c+v}\sqrt{1-\frac{v^2}{c^2}} = \frac{ct(A)(c-v)}{(c+v)(c-v)}\sqrt{1-\frac{v^2}{c^2}} = \frac{t(A)-\frac{v}{c^2}x(A)}{\sqrt{1-\frac{v^2}{c^2}}}$$

We calculate $x'(A)$ and $t'(A)$ in terms of frame K



Taking into account[1] that the length of the co-moving meter stick is only $\sqrt{1-\frac{v^2}{c^2}}$,

$$x(A) = t(A)v + x'(A)\sqrt{1-\frac{v^2}{c^2}}$$

and thus

$$x'(A) = \frac{x(A) - v\,t(A)}{\sqrt{1-\frac{v^2}{c^2}}}$$

[1]Note that in K', event A is not simultaneous with the event that the standard clock reading is $t(A)$. But distance, in any frame, is defined between simultaneous events. We will return to this issue.

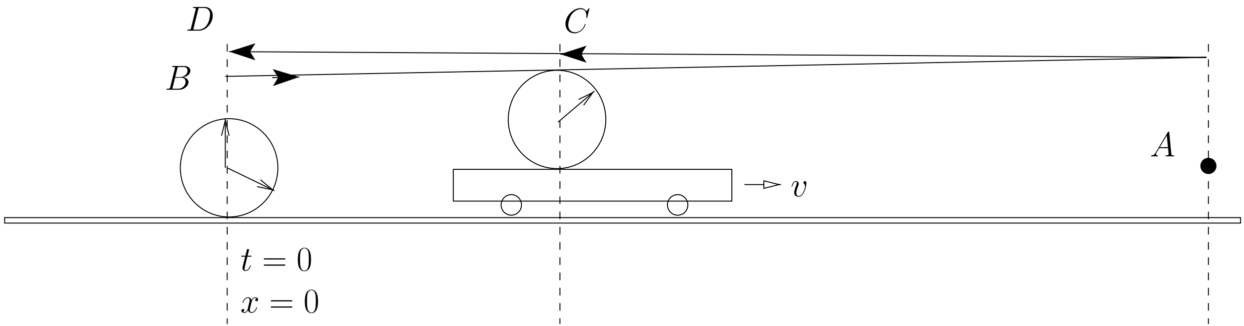
Lorentz transformations

$$
\begin{aligned}
t'(A) &= \frac{t(A) - \frac{v}{c^2}x(A)}{\sqrt{1 - \frac{v^2}{c^2}}} \\
x'(A) &= \frac{x(A) - v\,t(A)}{\sqrt{1 - \frac{v^2}{c^2}}}
\end{aligned}
$$

When $v/c \ll 1$ we get back the so-called Galilean transformations of classical physics:

$$
\begin{aligned}
t'(A) &= t(A) \\
x'(A) &= x(A) - v\,t(A)
\end{aligned}
$$

Relativity of simultaneity



- In K: A is simultaneous with the event that the "rest" clock's reading is $t(D)/2$
- In K': A is simultaneous with the event that the "moving" clock's reading is $t'(C)/2$
- But these two "clock" events are simultaneous neither in K nor in K'

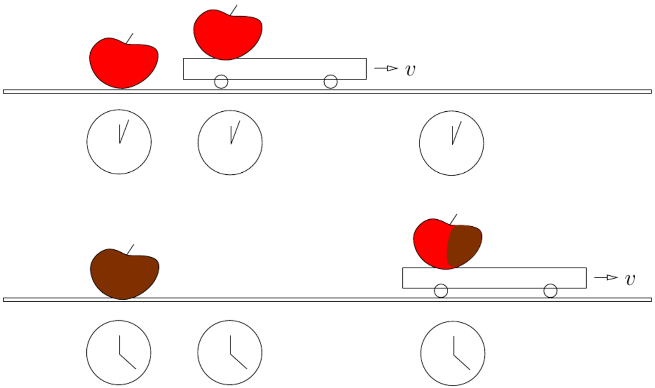
Relativity of simultaneity

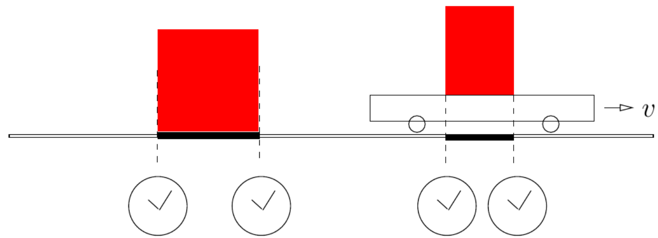
Consider two events, A and B, such that $t(A) = t(B)$. From the Lorentz transformations:

$$t'(A) - t'(B) = \frac{t(A) - \frac{v}{c^2}x(A)}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{t(B) - \frac{v}{c^2}x(B)}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{-\frac{v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}\left(x(A) - x(B)\right)$$

This isn't 0 unless $x(A) = x(B)$.

Principle of relativity





$$T_{moving} = \frac{T_{rest}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$L_{moving} = L_{rest}\sqrt{1 - \frac{v^2}{c^2}}$$

Constancy of speed of light across inertial frames

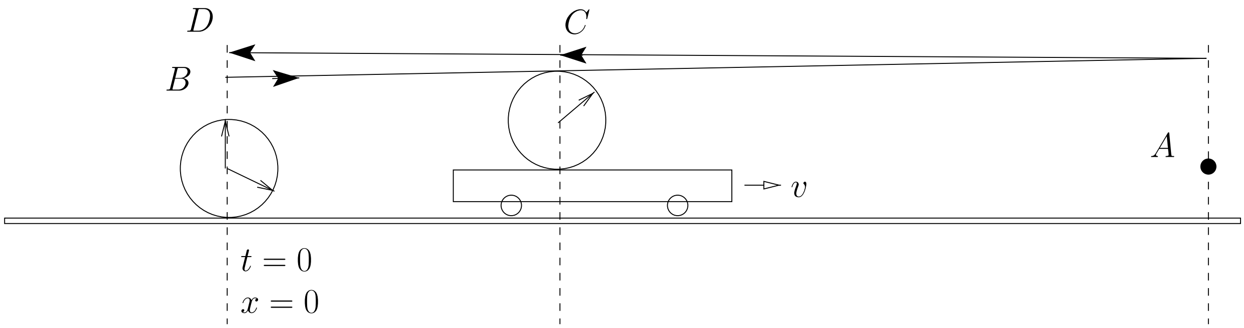
A light signal is emitted at event A, with $t(A) = 0$ and $x(A) = 0$. Then $t'(A) = 0$ and $x'(A) = 0$. The signal is absorbed at event B, thus $x(B) = ct(B)$. From the Lorentz transformations:

$$
\begin{aligned}
t'(B) &= \frac{t(B) - \frac{vct(B)}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \\
x'(B) &= \frac{ct(B) - v\,t(B)}{\sqrt{1 - \frac{v^2}{c^2}}}
\end{aligned}
$$

The speed of the light signal in K':

$$
v'(light) = \frac{x'(B)}{t'(B)} = \frac{\frac{ct(B) - v\,t(B)}{\sqrt{1 - \frac{v^2}{c^2}}}}{\frac{t(B) - \frac{vct(B)}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}} = c
$$

We calculate $x'(A)$ and $t'(A)$ in terms of frame K



We assume that the following hold in K:

1. Speed of light is the same in all directions and independent of its source
2. Time dilation and length contraction
3. Galilean kinematics (wrt. the space and time coordinates measured by rods and clocks at rest in K)

What we see is that the three conditions are *derivable* for K'. Consequently, the initially chosen inertial frame is not privileged and can be picked arbitrarily.

Minkowski geometry

Euclidean geometry in 3 dimensions

$$\left(\mathbb{R}^3, d\right)$$

Euclidean distance:

$$d\left(\mathbf{x}_1, \mathbf{x}_2\right) = \sqrt{\left(x_1 - x_2\right)^2 + \left(y_1 - y_2\right)^2 + \left(z_1 - z_2\right)^2}$$
$$\mathbf{x}_1 = \left(x_1, y_1, z_1\right), \mathbf{x}_2 = \left(x_2, y_2, z_2\right) \in \mathbb{R}^3$$

Euclidean geometry in n dimensions

$$(\mathbb{R}^n, d)$$

Euclidean distance:

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots + (u_n - v_n)^2}$$
$$\mathbf{u} = (u_1, ..., u_n), \mathbf{v} = (v_1, ..., v_n) \in \mathbb{R}^n$$

Scalar product in Euclidean geometry

In Euclidean geometry over $\mathbb{R}^3$ ($\mathbf{x}_1 = (x_1, y_1, z_1)\,, \mathbf{x}_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$):

- distance: $d\left(\mathbf{x}_1, \mathbf{x}_2\right) = \sqrt{\left(x_1 - x_2\right)^2 + \left(y_1 - y_2\right)^2 + \left(z_1 - z_2\right)^2}$
- scalar product: $p\left(\mathbf{x}_1, \mathbf{x}_2\right) = x_1 x_2 + y_1 y_2 + z_1 z_2$
- length: $l\left(\mathbf{x}\right) = \sqrt{p\left(\mathbf{x}, \mathbf{x}\right)} = \sqrt{x^2 + y^2 + z^2}$
- their link: $d\left(\mathbf{x}_1, \mathbf{x}_2\right) = l\left(\mathbf{x}_1 - \mathbf{x}_2\right) = \sqrt{p\left(\mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_2\right)}$
- orthogonality: $p\left(\mathbf{x}_1, \mathbf{x}_2\right) = 0$
- angle: $cos\alpha = \frac{p(\mathbf{x}_1, \mathbf{x}_2)}{l(\mathbf{x}_1) l(\mathbf{x}_2)}$

Euclidean geometry in 3 dimensions: version 2

$$\left(\mathbb{R}^3, p\right)$$

Euclidean scalar product:

$$p\left(\mathbf{x}_1, \mathbf{x}_2\right) = x_1 x_2 + y_1 y_2 + z_1 z_2$$

$$\mathbf{x}_1 = \left(x_1, y_1, z_1\right), \mathbf{x}_2 = \left(x_2, y_2, z_2\right) \in \mathbb{R}^3$$

Minkowski geometry

$$\left(\mathbb{R}^4, s\right)$$

Minkowski distance:

$$s\left(\mathbf{x}_1, \mathbf{x}_2\right) = \sqrt{\left(t_1 - t_2\right)^2 - \left(\left(x_1 - x_2\right)^2 + \left(y_1 - y_2\right)^2 + \left(z_1 - z_2\right)^2\right)}$$

$$\mathbf{x}_1 = \left(t_1, x_1, y_1, z_1\right), \mathbf{x}_2 = \left(t_2, x_2, y_2, z_2\right) \in \mathbb{R}^4$$

Minkowski distance and Lorentz product

In Minkowski geometry over $\mathbb{R}^4$ ($\mathbf{x}_1 = (t_1, x_1, y_1, z_1)\,, \mathbf{x}_2 = (t_2, x_2, y_2, z_2) \in \mathbb{R}^4$):

- Minkowski distance: $s\left(\mathbf{x}_1, \mathbf{x}_2\right) = \sqrt{\left(t_1 - t_2\right)^2 - \left(x_1 - x_2\right)^2 - \left(y_1 - y_2\right)^2 - \left(z_1 - z_2\right)^2}$
- Lorentz product: $\eta\left(\mathbf{x}_1, \mathbf{x}_2\right) = t_1 t_2 - x_1 x_2 - y_1 y_2 - z_1 z_2$
- Minkowski length: $\lambda\left(\mathbf{x}\right) = \sqrt{\eta\left(\mathbf{x}, \mathbf{x}\right)} = \sqrt{t^2 - x^2 - y^2 - z^2}$
- their link: $s\left(\mathbf{x}_1, \mathbf{x}_2\right) = \lambda\left(\mathbf{x}_1 - \mathbf{x}_2\right) = \sqrt{\eta\left(\mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_2\right)}$
- orthogonality: $\eta\left(\mathbf{x}_1, \mathbf{x}_2\right) = 0$

Minkowski geometry: version 2

$$\left(\mathbb{R}^4, \eta\right)$$

Lorentz product:

$$\eta\left(\mathbf{x}_1, \mathbf{x}_2\right) = t_1 t_2 - x_1 x_2 - y_1 y_2 - z_1 z_2$$

$$\mathbf{x}_1 = \left(t_1, x_1, y_1, z_1\right), \mathbf{x}_2 = \left(t_2, x_2, y_2, z_2\right) \in \mathbb{R}^4$$

Minkowski spacetime

Events are represented by their space and time coordinates (wrt. a given inertial frame) in a Minkowski geometry.[2] In two dimensions:

$$\eta\,(A,B) = c^2 t(A)t(B) - x(A)x(B)$$
$$s^2(A,B) = \eta\,(\mathbf{x}(B)-\mathbf{x}(A), \mathbf{x}(B)-\mathbf{x}(A)) = c^2\,(t(B)-t(A))^2 - (x(B)-x(A))^2$$

[2]One inserts c^2 in the definition of η for dimensional reasons, as well as to get out the correct relativistic formulas from it. Is is customary to choose convenient units in which $c = 1$.

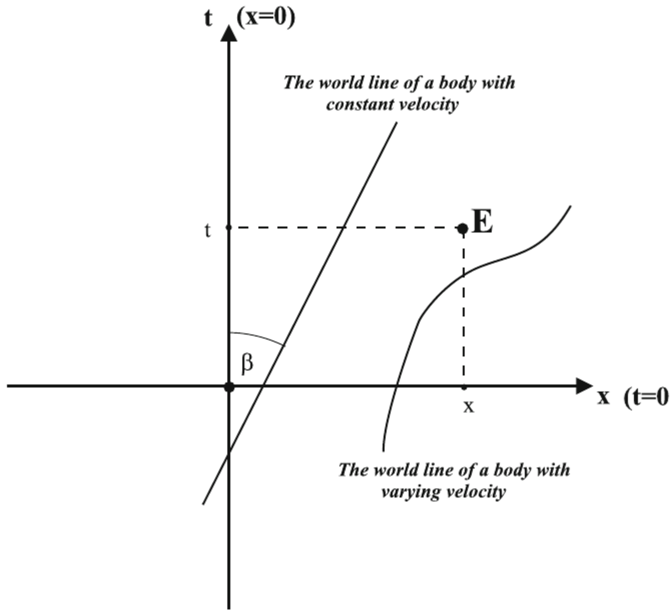
Minkowski spacetime

Events are represented by their space and time coordinates (wrt. a given inertial frame) in a Minkowski geometry. In two dimensions:

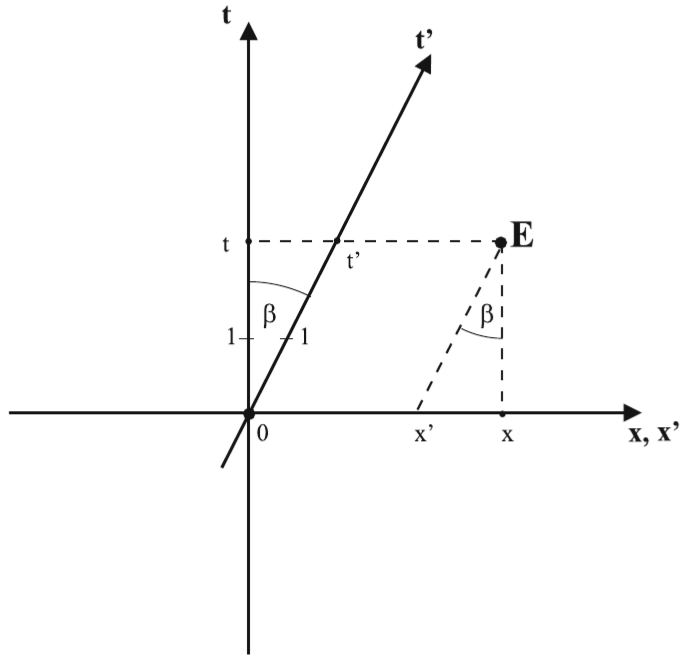
$$\eta\,(A, B) = c^2 t(A) t(B) - x(A) x(B)$$
$$s^2(A, B) = \eta\,(\mathbf{x}(B) - \mathbf{x}(A), \mathbf{x}(B) - \mathbf{x}(A)) = c^2\,(t(B) - t(A))^2 - (x(B) - x(A))^2$$

All of special relativistic physics can be expressed in terms of the Lorentz product structure.

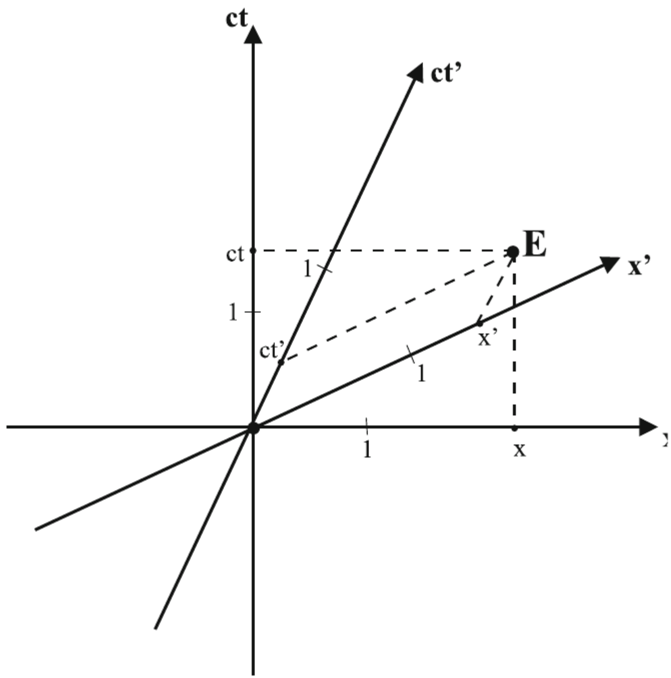
Spacetime diagram in classical physics



Spacetime coordinates in a moving inertial frame in classical physics



Spacetime coordinates in a moving inertial frame in relativity



Time and space axes of an inertial frame are Minkowski-orthogonal

- A vector that lies on the t'-axis of K' is of the form $\mathbf{x}_1 = (\tau, v\tau)$ for some $\tau \in \mathbb{R}$.
- A vector that lies on the x'-axis of K' has 0 t' coordinate. Hence, due to the Lorentz transformation, its t and x coordinate must satisfy

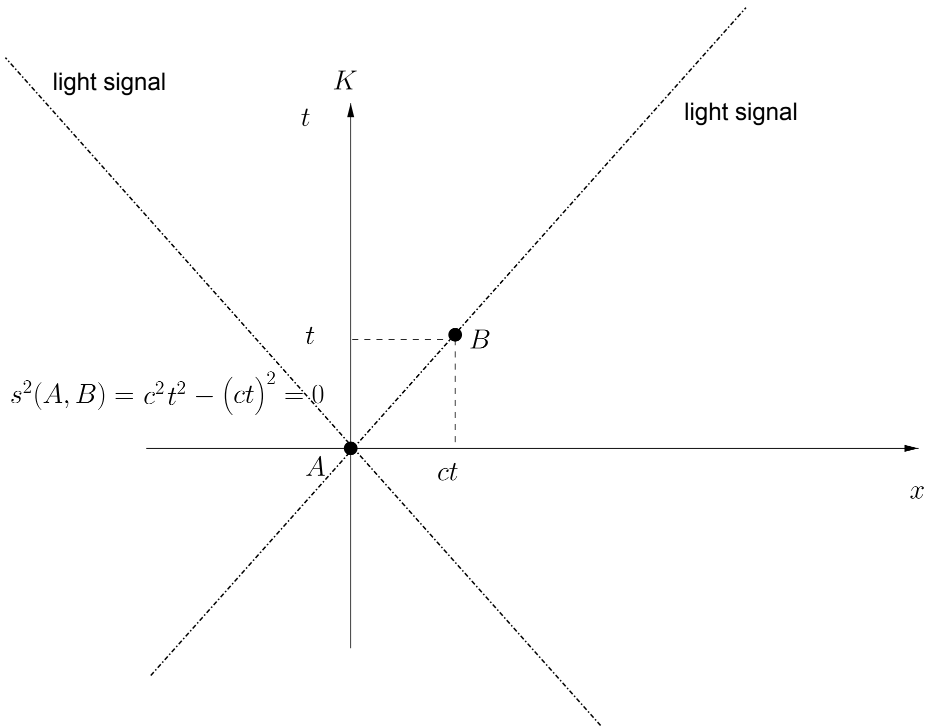
$$0 = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}}$$

  yielding $t = \frac{v}{c^2}x$. Thus, this vector must be of the from $\mathbf{x}_2 = \left(\frac{v}{c^2}\xi, \xi\right)$ for some $\xi \in \mathbb{R}$.

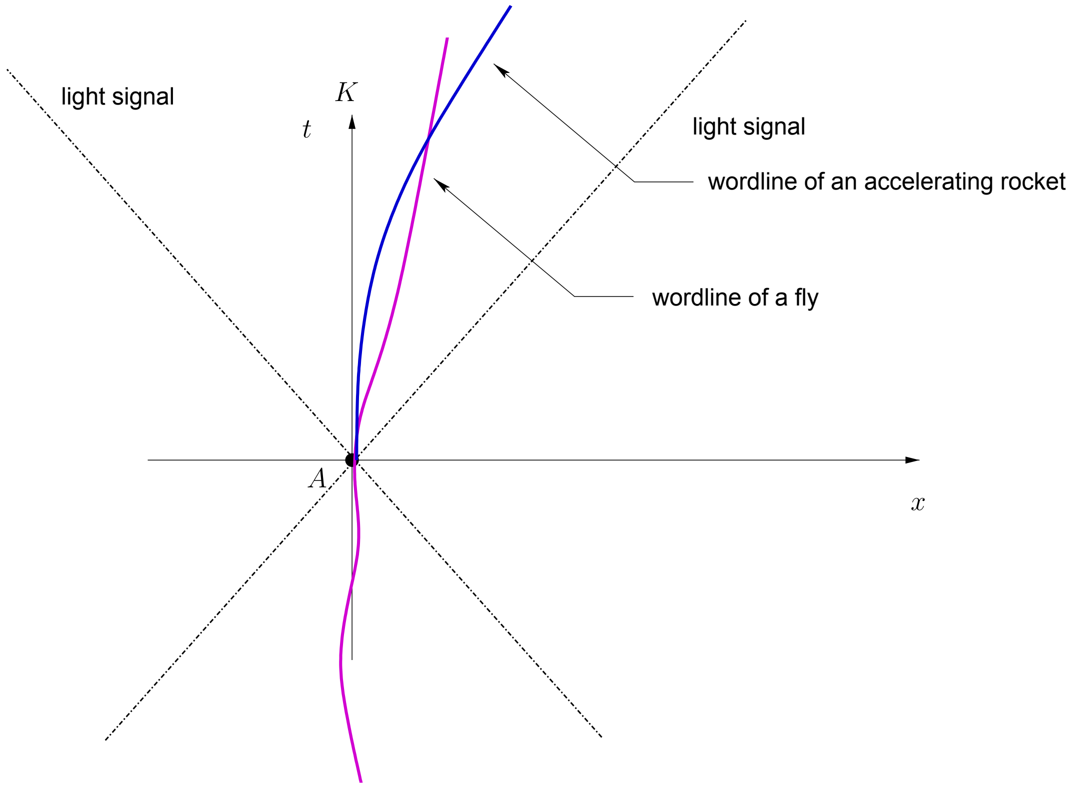
- Their Lorentz product:

$$\eta\left(\mathbf{x}_1, \mathbf{x}_2\right) = c^2 t_1 t_2 - x_1 x_2 = c^2 \tau \frac{v}{c^2} \xi - v\tau\xi = 0$$

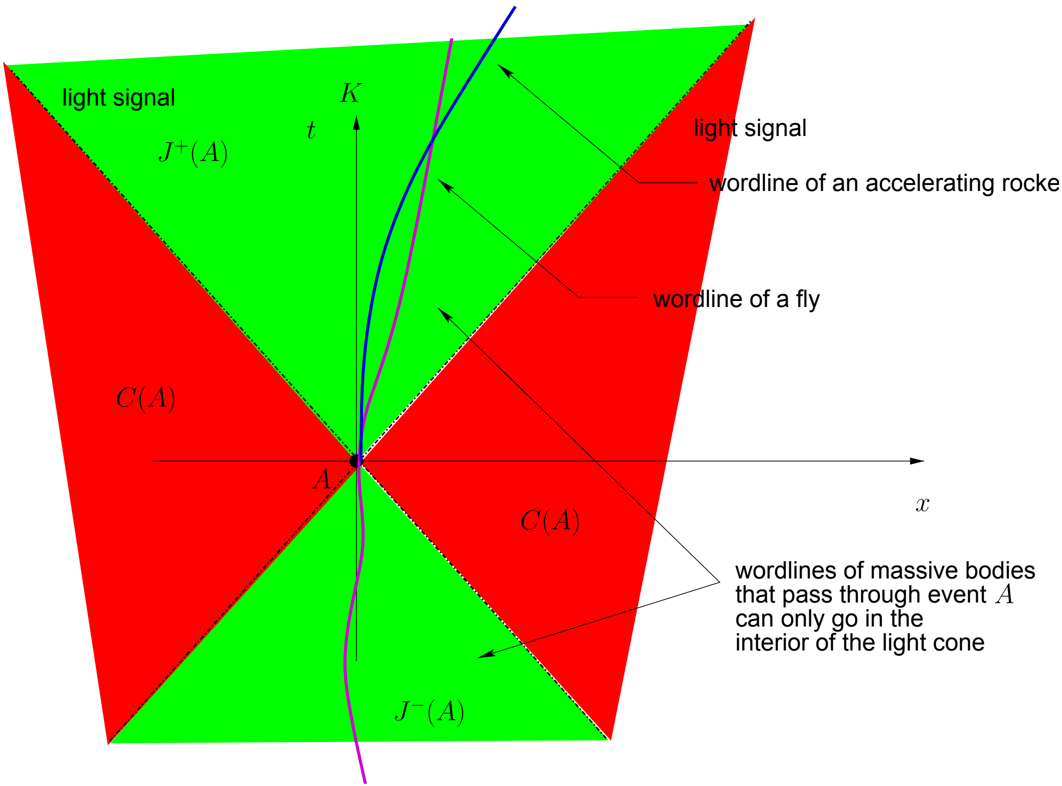
Path of light rays in spacetime



Path of massive bodies in spacetime



Light-cone structure



Light-cone structure

$\mathbf{x} \in \mathbb{R}^4$ is

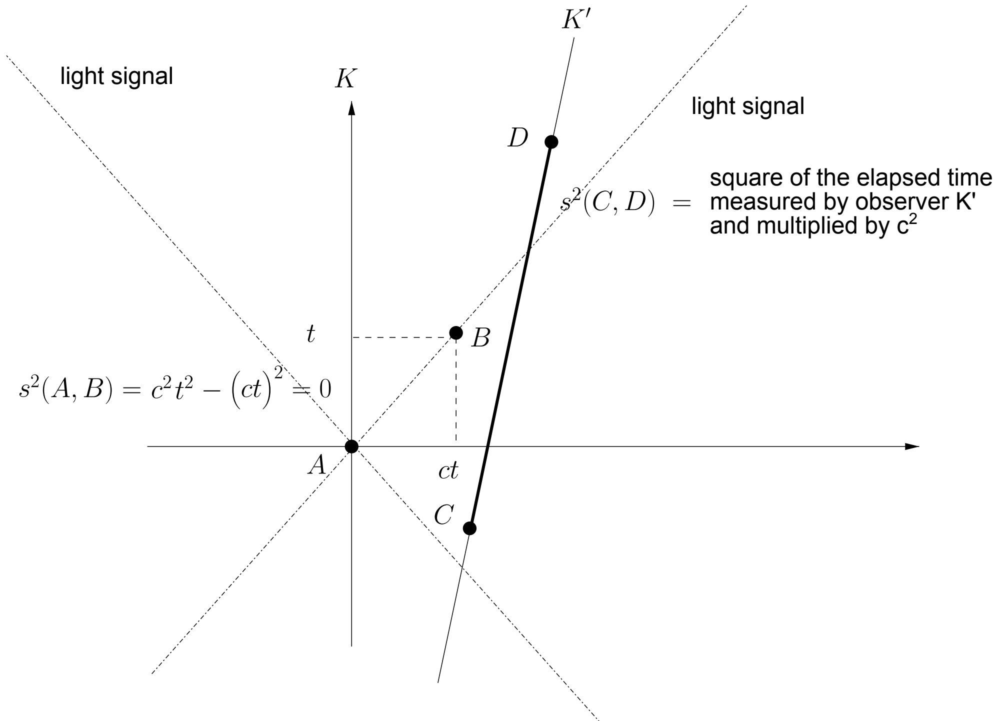
- time-like iff $\eta(\mathbf{x}, \mathbf{x}) > 0$
- light-like iff $\eta(\mathbf{x}, \mathbf{x}) = 0$
- space-like iff $\eta(\mathbf{x}, \mathbf{x}) < 0$
- (future-directed iff $t > 0$, past-directed iff $t < 0$)

Light-cone structure

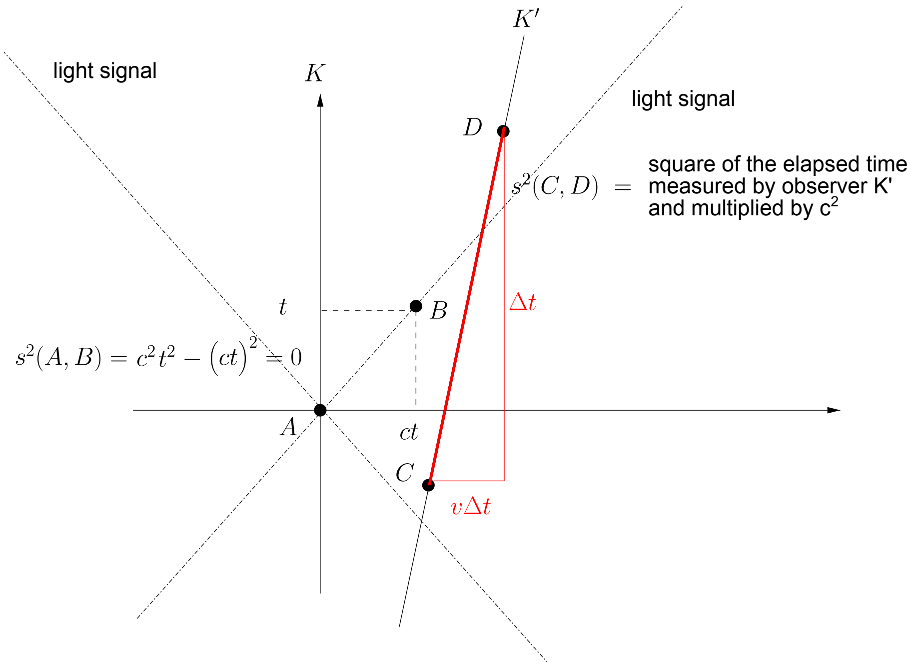
$\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^4$ are

- time-like separated iff $\eta\left(\mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_2\right) > 0$ ($s\left(\mathbf{x}_1, \mathbf{x}_2\right) > 0$)
- light-like separated iff $\eta\left(\mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_2\right) = 0$ ($s\left(\mathbf{x}_1, \mathbf{x}_2\right) = 0$)
- space-like separated iff $\eta\left(\mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_2\right) < 0$ ($s\left(\mathbf{x}_1, \mathbf{x}_2\right)$ is imaginary)
- (future light-cone, past light-cone)

Minkowski distance between time-like separated events is time measured by a clock moving inertially between those events

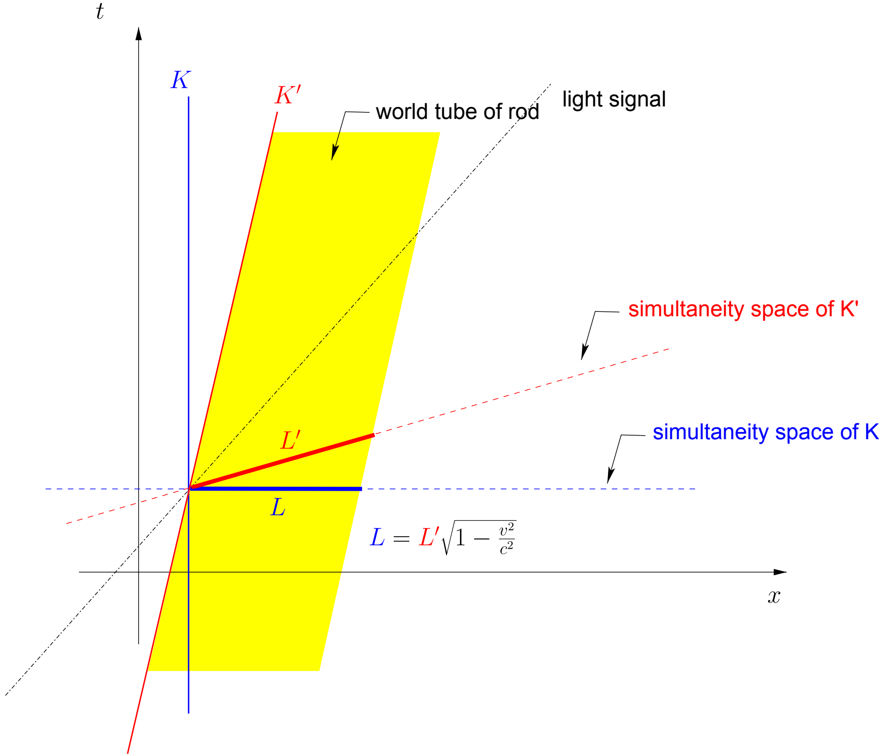


Minkowski distance between time-like separated events is time measured by a clock moving inertially between those events

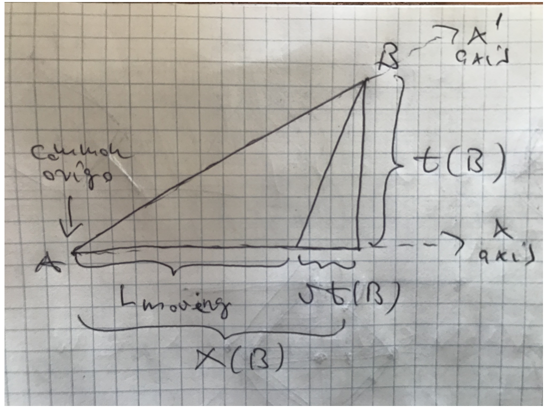


$$s^2(C,D) = c^2\Delta t^2 - v^2\Delta t^2 = c^2\Delta t^2\left(1 - \frac{v^2}{c^2}\right) = c^2 \times (\text{time measured by the moving clock})^2$$

Minkowski distance between space-like separated events is spatial distance measured with moving rods by an observer for whom those events are simultaneous



Minkowski distance between space-like separated events

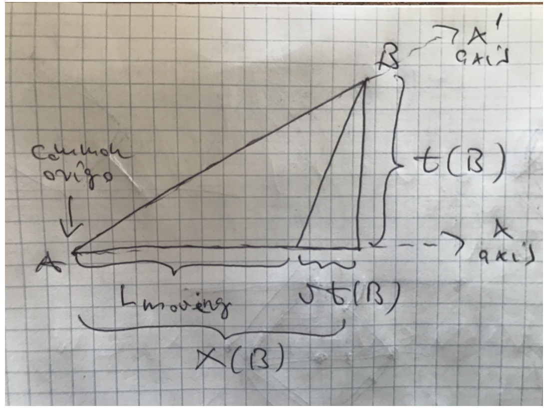


Events A and B are the two ends of the rod for observer K' at instant $t' = 0$. A occurs in the origin of the coordinate systems, so $t(A) = x(A) = 0$. Since $t'(B) = 0$, from the Lorentz transformation we have

$$0 = \frac{t(B) - \frac{v}{c^2} x(B)}{\sqrt{1 - \frac{v^2}{c^2}}}$$

yielding $t(B) = \frac{v}{c^2} x(B)$.

Minkowski distance between space-like separated events

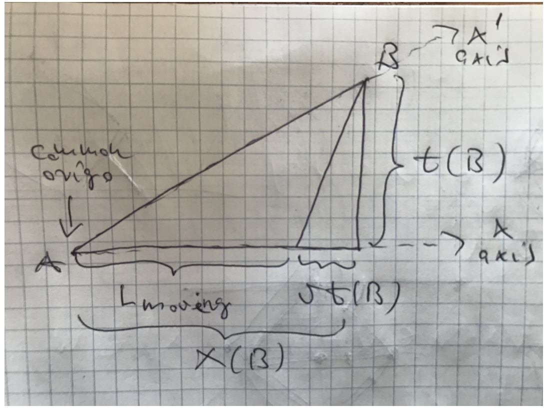


$$x\,(B) = L_{moving} + vt\,(B) = L_{moving} + v\frac{v}{c^2}x\,(B)$$

which yields

$$x\,(B) = \frac{L_{moving}}{1 - \frac{v^2}{c^2}}$$

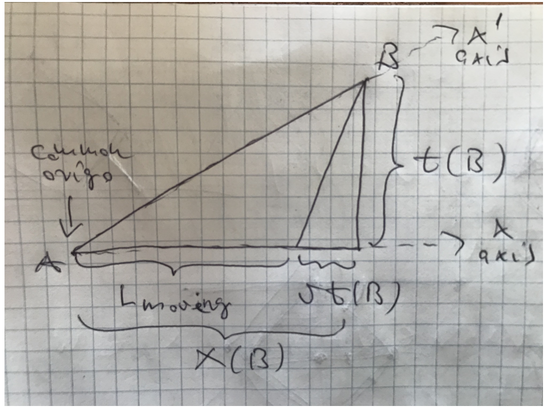
Minkowski distance between space-like separated events



Finally, due to length contraction the connection between the length of the moving rod as measured in K and K' is the following:

$$L_{moving} = L'_{moving}\sqrt{1-\frac{v^2}{c^2}}$$

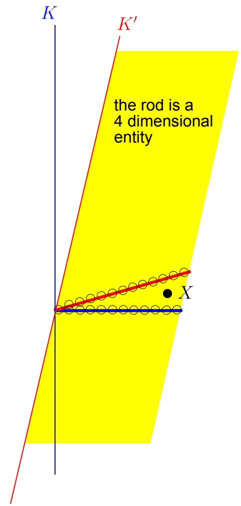
Minkowski distance between space-like separated events



Putting all this together, we have

$$\begin{aligned} s^2(A,B) &= c^2 t\,(B)^2 - x\,(B)^2 = c^2\left(\frac{v}{c^2}x\,(B)\right)^2 - x\,(B)^2 = -x\,(B)^2\left(1-\frac{v^2}{c^2}\right) \\ &= -\frac{L_{moving}^2}{1-\frac{v^2}{c^2}} = -\left(L'_{moving}\right)^2 = -\left(\text{distance measured by moving observer}\right)^2 \end{aligned}$$

Physical objects are 4 dimensional entities



- At the moment of their meeting, for the two observers the rod is a collection of different events.
- Physical objects are 4 dimensional entities. Only things in 4 dimensions are real; 3 dimensional space and time are just perspectives of the 4 dimensional reality.

Lorentz transformation preserves the Lorentz product

Lorentz product in 2D:

$$\eta\left(\mathbf{x}_1, \mathbf{x}_2\right) = c^2 t_1 t_2 - x_1 x_2 \qquad \mathbf{x}_1 = \left(t_1, x_1\right), \mathbf{x}_2 = \left(t_2, x_2\right) \in \mathbb{R}^2$$

Lorentz product of a Lorentz transformed pair of vectors:

$$\begin{aligned}
\eta\left(\mathbf{x}_1', \mathbf{x}_2'\right) &= c^2 t_1' t_2' - x_1' x_2' = c^2 \frac{t_1 - \frac{v}{c^2} x_1}{\sqrt{1-\frac{v^2}{c^2}}} \frac{t_2 - \frac{v}{c^2} x_2}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{x_1 - v t_1}{\sqrt{1-\frac{v^2}{c^2}}} \frac{x_2 - v t_2}{\sqrt{1-\frac{v^2}{c^2}}} \\
&= \frac{1}{1-\frac{v^2}{c^2}} \left(c^2 t_1 t_2 - v t_1 x_2 - v x_1 t_2 + \frac{v^2}{c^2} x_1 x_2 - x_1 x_2 + v x_1 t_2 + v t_1 x_2 - v^2 t_1 t_2 \right) \\
&= c^2 t_1 t_2 - x_1 x_2 = \eta\left(\mathbf{x}_1, \mathbf{x}_2\right)
\end{aligned}$$

Minkowski spacetime

Events are represented by their space and time coordinates (wrt. a given inertial frame) in a Minkowski geometry. In two dimensions:

$$\eta\,(A,B) = c^2 t(A)t(B) - x(A)x(B)$$
$$s^2(A,B) = \eta\,(\mathbf{x}(B) - \mathbf{x}(A), \mathbf{x}(B) - \mathbf{x}(A)) = c^2\,(t(B) - t(A))^2 - (x(B) - x(A))^2$$

All of special relativistic physics can be expressed in terms of the Lorentz product structure.

Minkowski spacetime

Events are represented by their space and time coordinates (wrt. ANY given inertial frame) in a Minkowski geometry. In two dimensions:

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All of special relativistic physics can be expressed in terms of the Lorentz product structure.

Euclidean geometry in 3 dimensions: version 2

$$\left(\mathbb{R}^3, p\right)$$

Euclidean scalar product:

$$p\left(\mathbf{x}_1, \mathbf{x}_2\right) = x_1 x_2 + y_1 y_2 + z_1 z_2$$
$$\mathbf{x}_1 = \left(x_1, y_1, z_1\right), \mathbf{x}_2 = \left(x_2, y_2, z_2\right) \in \mathbb{R}^3$$

Euclidean geometry in 3 dimensions: version 3

$$(V, p)$$

where V is a 3 dimensional vector space and p is a Euclidean scalar product on it, that is:

1. For all $\mathbf{u}, \mathbf{v} \in V$, $p(\mathbf{u}, \mathbf{v}) = p(\mathbf{v}, \mathbf{u})$
2. For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $p(\mathbf{u}, \mathbf{v} + \mathbf{w}) = p(\mathbf{u}, \mathbf{v}) + p(\mathbf{u}, \mathbf{w})$
3. For all $\alpha \in \mathbb{R}$ and all $\mathbf{u}, \mathbf{v} \in V$, $p(\mathbf{u}, \alpha\mathbf{v}) = \alpha p(\mathbf{u}, \mathbf{v})$
4. For all non-zero $\mathbf{u} \in V$, there is a $\mathbf{v} \in V$ such that $p(\mathbf{u}, \mathbf{v}) \neq 0$
5. The signature of p is $(3, 0)$

Orthonormal basis

Let vectors $\mathbf{v}_1, ..., \mathbf{v}_n \in V$ form a basis of V. That is,

1) $\mathbf{v}_1, ..., \mathbf{v}_n$ are linearly independent,
2) any other vector $\mathbf{v} \in V$ is expressible as a linear combination of $\mathbf{v}_1, ..., \mathbf{v}_n$.

We say that $\mathbf{v}_1, ..., \mathbf{v}_n$ is an orthonormal basis of V if, for all $\mathbf{v}_i, \mathbf{v}_k$

- if $\mathbf{v}_i \neq \mathbf{v}_k$ then $p\left(\mathbf{v}_i, \mathbf{v}_k\right) = 0$
- $p\left(\mathbf{v}_i, \mathbf{v}_i\right)^2 = 1$

Signature

The signature of p is a pair of non-negative integers (n^+, n^-) where

$$\begin{aligned} n^+ &= \text{number of vectors } \mathbf{v}_i \text{ in an orthonormal basis with } p\left(\mathbf{v}_i, \mathbf{v}_i\right) = 1 \\ n^- &= \text{number of vectors } \mathbf{v}_i \text{ in an orthonormal basis with } p\left(\mathbf{v}_i, \mathbf{v}_i\right) = -1 \end{aligned}$$

This definition makes sense as one can prove that numbers n^+ and n^- are the same for any orthonormal basis.

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This definition makes sense as one can prove that numbers n^+ and n^- are the same for any orthonormal basis.

If all vectors of an orthonormal basis are of positive square length, then p a is Euclidean scalar product.

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5. The signature of p is $(3, 0)$

This abstract characterization makes sense as every 3 dimensional Euclidean space (V, p) is *isomorphic*[3] to $(\mathbb{R}^3, p_{\mathbb{R}^3})$, the Euclidean space à la version 2.

[3] This means that there exists a bijection $i : V \to \mathbb{R}^3$ that preserves all the operations. E.g. we have $p(\mathbf{u}, \mathbf{v}) = p_{\mathbb{R}^3}(i(\mathbf{u}), i(\mathbf{v}))$ for all $\mathbf{u}, \mathbf{v} \in V$.

Minkowski geometry: version 2

$$\left(\mathbb{R}^4, \eta\right)$$

Lorentz product:

$$\eta\left(\mathbf{x}_1, \mathbf{x}_2\right) = t_1 t_2 - x_1 x_2 - y_1 y_2 - z_1 z_2$$

$$\mathbf{x}_1 = \left(t_1, x_1, y_1, z_1\right), \mathbf{x}_2 = \left(t_2, x_2, y_2, z_2\right) \in \mathbb{R}^4$$

Minkowski geometry: version 3

$$(V, \eta)$$

where V is a 4 dimensional vector space and η is a Minkowskian scalar product (or Lorentz product) on it, that is:

1. For all $\mathbf{u}, \mathbf{v} \in V$, $\eta(\mathbf{u}, \mathbf{v}) = \eta(\mathbf{v}, \mathbf{u})$
2. For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $\eta(\mathbf{u}, \mathbf{v} + \mathbf{w}) = \eta(\mathbf{u}, \mathbf{v}) + \eta(\mathbf{u}, \mathbf{w})$
3. For all $\alpha \in \mathbb{R}$ and all $\mathbf{u}, \mathbf{v} \in V$, $\eta(\mathbf{u}, \alpha\mathbf{v}) = \alpha\eta(\mathbf{u}, \mathbf{v})$
4. For all non-zero $\mathbf{u} \in V$, there is a $\mathbf{v} \in V$ such that $\eta(\mathbf{u}, \mathbf{v}) \neq 0$
5. The signature of η is $(1, 3)$

Minkowski geometry: version 3

$$(V, \eta)$$

where V is a 4 dimensional vector space and η is a Minkowskian scalar product (or Lorentz product) on it, that is:

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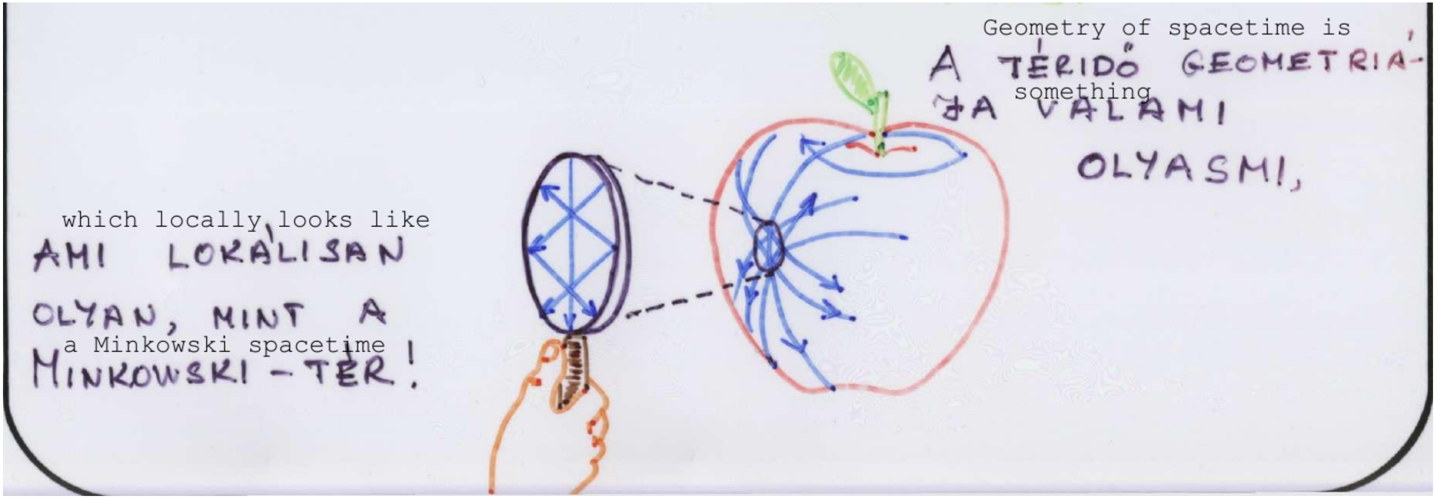
5. The signature of η is $(1, 3)$

This abstract characterization makes sense as every Minkowski space (V, η) is *isomorphic*[4] to $(\mathbb{R}^4, \eta_{\mathbb{R}^4})$, the Minkowski space à la version 2.

[4] This means that there exists a bijection $i : V \rightarrow \mathbb{R}^4$ that preserves all the operations. E.g. we have $\eta(\mathbf{u}, \mathbf{v}) = \eta_{\mathbb{R}^4}(i(\mathbf{u}), i(\mathbf{v}))$ for all $\mathbf{u}, \mathbf{v} \in V$.

Spacetime is locally Minkowski-like



Affine space

Vector spaces have a distinguished $\mathbf{0}$ element. Thus they are not appropriate for representing homogeneous spacetime structure. An "affine space" can be thought of as a vector space with the $\mathbf{0}$ element washed out. More precisely, we have the following definition.

> An affine space is a structure
>
> $$(A, V, +)$$
>
> where
>
> - A is a non-empty set
> - V is a vector space
> - $+ : A \times V \to A$ is a map satisfying the following conditions:
>   1. For all $p, q \in A$, there is a unique $\mathbf{u} \in V$ such that $q = p + \mathbf{u}$
>   2. For all $p \in A$, and all $\mathbf{u}, \mathbf{v} \in V$, $(p + \mathbf{u}) + \mathbf{v} = p + (\mathbf{u} + \mathbf{v})$

Euclidean geometry in 3 dimensions: version 4

$$(A, V, +, p)$$

where

- V is a 3 dimensional vector space
- $(A, V, +)$ is an affine space
- p is a Euclidean scalar product on V

Minkowski geometry: version 4

$$(A, V, +, \eta)$$

where

- V is a 4 dimensional vector space
- $(A, V, +)$ is an affine space
- η is a Minkowskian scalar product on V

Flat geometries

$$
\begin{tikzcd}
 & \text{affine spaces} \arrow[d] & \\
 & \text{metric affine spaces} \arrow[dl] \arrow[dr] & \\
\text{Euclidean spaces} & & \text{Minkowskian spaces}
\end{tikzcd}
$$

Note on methodology

The geometric approach of special relativity starts from the abstract definition of Minkowski space, and add interpretative principles to it that connect geometry with physics. Such principles are the following:

- Time-like curves represent the spacetime trajectories of massive point particles, i.e., point particles with non-zero mass.
- Time-like lines represent the spacetime trajectories of free massive point particles, i.e., massive point particles that are not subject to any force.
- Light-like lines represent the spacetime trajectories of light rays.
- Minkowski distance between time-like separated events is time measured by a clock moving inertially between those events.
- Minkowski distance between space-like separated events is spatial distance measured with moving rods by an observer for whom those events are simultaneous.

Note on methodology

One can then *derive* the paradigmatic special relativistic effects from the geometry plus these interpretative principles.

By contrast, in our approach (sometimes termed as the dynamical approach), these principles were *consequences* of the definition of the Lorentz product in the coordinates of any given inertial frame, in conjunction with the special relativistic phenomena we started out with.