Conceptual introduction to spacetime geometry

15 February 2024

Today

Vectors

Vector space

A real vector space is a set V whose elements can be added and multiplied by real numbers in harmony with to the usual rules of addition and multiplication.

Linear combination

$$\sum_{i=1}^{n} \alpha_i \mathbf{v}_i \qquad \alpha_i \in \mathbb{R}, \mathbf{v}_i \in V$$

Linear independence

Vectors $\mathbf{v}_1, ..., \mathbf{v}_n \in V$ are linearly independent if non of them is expressible as a linear combination of the others. That is, for any $k = 1, ..., n$ there are no $\alpha_i \in \mathbb{R}, i = 1, ..., n, i \neq k$ such that

$$\mathbf{v}_k = \sum_{\substack{i=1 \\ i \neq k}}^{n} \alpha_i \mathbf{v}_i$$

Basis

Vectors $\mathbf{v}_1, ..., \mathbf{v}_n \in V$ form a basis of V iff

1) $\mathbf{v}_1, ..., \mathbf{v}_n$ are linearly independent,
2) any other vector $\mathbf{v} \in V$ is expressible as a linear combination of $\mathbf{v}_1, ..., \mathbf{v}_n$.

Dimension

All basis has the same number of basis vectors in it and this number is called the dimension of V.

Linear transformations

Let U and V be vector spaces. A map $A : U \to V$ is linear iff for all $\mathbf{u}_1, ..., \mathbf{u}_n \in V, \alpha_1, ..., \alpha_n \in \mathbb{R}$

$$A\left(\sum_{i=1}^{n} \alpha_i \mathbf{u}_i\right) = \sum_{i=1}^{n} \alpha_i A\left(\mathbf{u}_i\right)$$

The set of all linear maps from U to V forms a vector space.

Coordinatization

A basis $\mathbf{v}_1, ..., \mathbf{v}_n \in V$ induces a linear bijection between V and $\mathbb{R}^n$ which is called a coordinatization. In other words, relative to a given basis $\mathbf{v}_1, ..., \mathbf{v}_n \in V$, vectors of V can be represented by vectors of $\mathbb{R}^n$.

Matrix representation

$\mathbb{R}^m \to \mathbb{R}^n$ linear maps can be represented as matrix products.

Therefore, relative to given bases in U and V, linear maps $U \to V$ can also be represented as matrix products.

Dual space

The set of linear maps from V to $\mathbb{R}$ is called the dual space of V, and denoted by V^*. The elements of V^* are called covectors (or 1-forms).

V^* is a vector space.

Dual basis

Let $\mathbf{v}_1, ..., \mathbf{v}_n \in V$ be a basis of V. Then covectors $\mathbf{p}^1, ..., \mathbf{p}^n \in V^*$ defined by

$$\mathbf{p}^i\left(\mathbf{v}_j\right) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

form a basis of V^*.

Hence, the dimension of V^* is equal to the dimension of V.

Tensors

A map

$$T : V^r \times (V^*)^s \to \mathbb{R}, \left(\mathbf{v}_1, ..., \mathbf{v}_r, \mathbf{p}^1, ..., \mathbf{p}^s\right) \mapsto T\left(\mathbf{v}_1, ..., \mathbf{v}_r, \mathbf{p}^1, ..., \mathbf{p}^s\right) \in \mathbb{R}$$

that is linear in all of its arguments is called a tensor of type (r, s).

The set of tensors of of type (r, s) forms a vector space.

Tensor product

Let $\mathbf{p}, \mathbf{q} \in V^*$.

$$\mathbf{p} \otimes \mathbf{q} : V \times V \to \mathbb{R}, (\mathbf{p} \otimes \mathbf{q})(\mathbf{u}, \mathbf{v}) \mapsto \mathbf{p}(\mathbf{u})\, \mathbf{q}(\mathbf{v})$$

is a $(2, 0)$-type tensor, which is called the tensor product of $\mathbf{p}$ and $\mathbf{q}$.

Scalar product

A map $s : V \times V \to \mathbb{R}$ is called a scalar product iff s is

- linear in both arguments
- symmetric, that is $s\left(\mathbf{v}_1, \boldsymbol{v}_2\right) = s\left(\mathbf{v}_2, \boldsymbol{v}_1\right)$
- $s\left(\mathbf{v}, \boldsymbol{v}\right) \geq 0$ and $s\left(\mathbf{v}, \boldsymbol{v}\right) = 0 \Leftrightarrow \boldsymbol{v} = 0$

Pseudo scalar product

A map $s : V \times V \to \mathbb{R}$ is called a pseudo scalar product iff s is

- linear in both arguments
- symmetric, that is $s(\mathbf{v}_1, \boldsymbol{v}_2) = s(\mathbf{v}_2, \boldsymbol{v}_1)$
- nondegenerate, that is, if $s(\mathbf{u}, \boldsymbol{v}) = 0$ for all $\mathbf{u} \in V$ then $\boldsymbol{v} = 0$.