Physicalist Metaphysical Foundation of Mathematics

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Part I

Physicalist Metaphysical Foundation of Mathematics

Empiricism: Genuine information about the world must be acquired by *a posteriori* means.	+	**Physicalist account of the mental**: Experiencing itself, as any other mental phenomena, including the mental processing the experiences, can be wholly explained in terms of physical properties, states, and events in the physical world.

Physicalist Metaphysical Foundation of Mathematics

- If—as I will claim—mathematics is a system of meaningless signs, then any claim *about* mathematics must be *external* to mathematics.
- A big part of the corpus of contemporary mathematics is *not invariant* over the possible philosophical positions. *If Gödel is right* by saying that

  > after sufficient clarification of the concepts in question it will be possible to conduct these discussions **with mathematical rigor** and that the result then will be that ... **the platonistic view is the only one tenable**

  then it follows that *one cannot avoid questioning certain parts of mathematics from an empiricist/physicalist standpoint.*
- I do not suggest to question in any way any of the *strict* formal derivations of mathematics. The philosophically non-invariant parts of mathematics refer to those claims of mathematicians that are not based on strict proofs but based on some naive theories and intuition.

The essential difference between mathematical truth and semantical truth in a scientific theory describing something in the world

A **physical theory** P is a formal system L + a semantics S pointing to the empirical world. In the construction of the formal system L one can *employ* previously prepared formal systems which come from mathematics and/or logic. That is, in general, L is a (first-order) system with

- some *logical axioms and the derivation rules* (usually the first-order predicate calculus with identity)
- the axioms of certain *mathematical theories*
- some *physical axioms*.

A sentence A in physical theory P can be true in two different senses:

Truth$_1$: A is a theorem of L, that is, $\vdash_L A$ (which is a mathematical truth within the formal system L, a fact of the formal system L).

Truth$_2$: According to the semantics S, A refers to an empirical fact (about the physical system described by P).

Example:

The electric field strength of a point charge is $\frac{kQ}{r^2}$

is a theorem of Maxwell's electrodynamics—one can derive the corresponding formal expression from the Maxwell equations. (This is a fact of the formal system L.) On the other hand, according to the semantics relating the symbols of the Maxwell theory to the empirical terms, this sentence corresponds to an empirical fact (about the point charges).

Truth$_1$ and Truth$_2$ are independent concepts – one does not automatically imply the other

Assume that

- Γ is a set of true$_2$ sentences in L, i.e., each sentence in Γ refers to an empirical fact
- and $\Gamma \vdash_L A$

It does not automatically follow that A is true$_2$. **Whether A is true$_2$ is again an empirical question:**

If so, then it is new empirically obtained information about the world, confirming the validity of the ***whole*** physical theory $P = L + S$.

If not, then this information disconfirms the physical theory, ***as a whole***. That is to say, one has to think about *revising* ***one of the*** *constituents* of P.

NB. Here we can see how Quine's *semantic* holism works. The unit of meaning is not the single sentence, but systems of sentences or even the whole of language. However, the empirical disconfirmation of a physical theory, in which, for example, the Euclidean geometry is applied, can disconfirm the *applicability* of Euclidean geometry in the physical theory in question, *but it leaves Euclidean geometry itself intact*. Here we can observe how Quine's *confirmational* holism fails.

Mathematical objects have no meanings

Thesis **Mathematical "statements" are formulas of a formal language. They are not linguistic objects, consequently they *carry no meanings and Tarskian truths.***

Arguments

I apply the VERIFIABILITY THEORY OF MEANING in the following very weak sense: in order to determine the meaning of a scientific (mathematics included) statement we follow up how the statement in question is, in principle, confirmed or discomfirmed.

1.

Let us first consider **physical realism**. If a mathematical proposition is an assertion about the physical world, then its truth-condition would be the correspondence with the physical facts. That is to say, *to decide whether a mathematical proposition is true or not, we would have to investigate the state of affairs in the physical world.*

But, in order to confirm or disconfirm a mathematical statement, we never see any reference to the physical world!

- Have you ever seen a mathematician in the *laboratory*?
- *What kind of experiment* should be performed in the laboratory in order to decide whether the group-theoretical statement $e(ee) = e$ is true or not?
- In the weaker version of physical realism, *not all mathematical propositions have meanings, only the most elementary ones*. For example, the axioms of set theory are often claimed (P. Maddy) to express evident truths about the "real sets" consisting of real objects of the world. But,
  - But, what about the *axiom of choice*? It seems to express an elementary feature of the "real sets", we are yet baffled about whether it should be added to the list of axioms or not, depending on some delicate mathematical considerations.
  - It would be difficult to tell what feature of "real sets" is reflected in the *continuum hypothesis*.
  - Such an unquestioned axiom as the *axiom of infinity* does not reflect any feature of "real sets" in the physical world. It rather reflects the wish of the set-theorist to have infinite sets.

    The axioms of the most fundamental mathematical structures are chosen on the basis of inherent mathematical reasonings, rather than on the basis of physical facts.

  - Assume, for the sake of the argument, that a kind of Truth_2 can be assigned to the axioms. *It would not follow that this Truth_2 can automatically be transmitted* to the more complex mathematical propositions derived from the axioms.
  - On the other hand, the claim that sets and some elementary statements of set theory reflect physical facts is a *description of the physical world*, that is to say, it is a physical theory. For example, if Jauch and Piron's description of the properties of physical objects is correct, then the property lattice of a quantum system does not satisfy the distributive law

$$\forall A \forall B \forall C \left[A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \right] \quad (1)$$

consequently, *the properties of a quantum mechanical object cannot be identified with sets*. Thus, contrary to our intuitions about "properties", "classes", "having a property", "belonging to a class", etc., such a simple true_1 set-theoretic formula as (1) fails to be satisfied by the properties of a quantum mechanical object, that is to say, in this case it is not true_2.

2.

Everything I told about physical realism, *mutatis mutandis*, can be applied to **intuitionism** and **platonic realism**.

physical world	⇒	world of mental/psychological phenomena or platonic world
laboratory experiment	⇒	other means by which knowledge can be obtained about these other realms

But, *in order to confirm or disconfirm a mathematical statement, the mathematician never refers to these worlds and the corresponding epistemic faculties!*

In Dummett's words:

> [T]here is nothing in mathematics that could be described as inference to the best explanation. Above all, we do not seek, in order to refute or confirm a hypothesis, a means of refining our intuitive faculties, as astronomers seek to improve their instruments. Rather, **if we suppose the hypothesis true, we seek for a *proof* of it**, and it remains a mere hypothesis, whose assertion would therefore be unwarranted, until we find one.

The physicalist ontology of formal systems

It is a common belief that philosophy of mathematics must take account of our impression that mathematical truth is a reflection of fact. As Hardy expresses this constraint,

> [N]o philosophy can possibly be sympathetic to a mathematician which does not admit, in one manner or the other, the immutable and unconditional validity of mathematical truth. Mathematical theorems are true or false; their truth or falsity is absolute and independent of our knowledge of them. In *some* sense, mathematical truth is a part of objective reality.

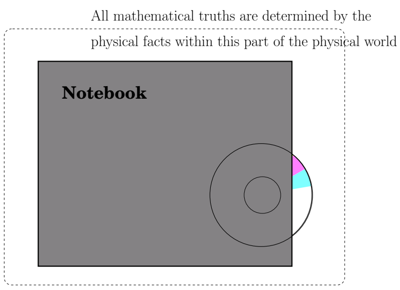
Now we determine what this objective reality actually is.

Thesis:

The objective fact expressed by a mathematical proposition is a fact of a particular part of the physical world: it is a fact of the formal system itself, that is, a fact about the physical system consisting of the signs and the mechanical rules according to which the signs can be combined.

Arguments

- Taking into account that the only means of obtaining reliable knowledge about this fact is mathematical proof, ***it must be a fact of the realm inside of the scope of formal derivations.***



- Sometimes it is argued that symbolism is merely a "convenient shorthand writing" to register the results obtained by *thinking*. However, in the contemporary mathematics there are derivations which are ***not surveyable by the human mind.*** (the proof of the four-color theorem) Sometimes even ***the theorem obtained through the derivation process is not surveyable***. (symbolic computer language manipulations)

- Sometimes one executes simple formal derivations also **in the head**. However, *from the point of view of the physicalist interpretation of mind* this case of formal manipulation **does not principally differ** from any other cases of derivation processes.

'$3+2=5$'	This *is not* a linguistic object!
actually means that	*the usual formalist step*
'formula $3+2=5$ derives from the formulas called the axioms of arithmetics'	This *is* a linguistic object!
which is nothing but	*the physicalist step*
the assertion that there exists a proof-process in the formal system called arithmetic, the result of which is the formula $3+2=5$	This is a usual scientific assertion, just as the assertion of the chemist about the existence of the process $2H_2 + O_2 \rightarrow 2H_2O$

In this way, a mathematical truth **has contingent factual content, as any similar scientific assertion**. It is

- expressing **objective fact** of the physical world
- **synthetic**
- ***a posteriori***
- **not necessary** and **not certain**
- **true before anybody can prove it**

Abstraction is a move from the concrete to the concrete

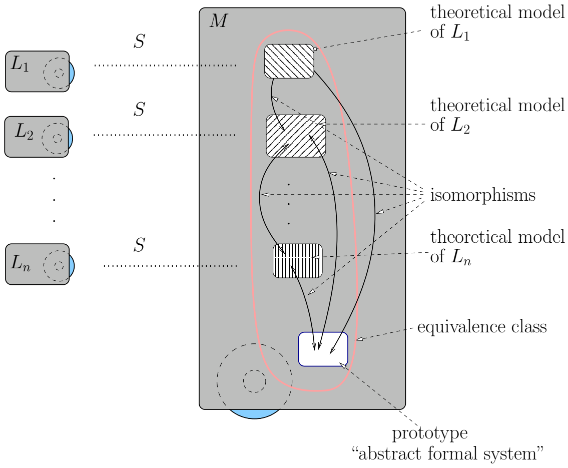
Sometimes we find the following ambivalent views in the formalist school. Curry writes:

> **...** in order to think of a formal system at all we must think of it as represented somehow. But when we think of it *as* formal system we abstract from all properties peculiar to the representation**.**

What does such an "abstraction" actually mean?

What do we obtain if we abstract from some unimportant, peculiar properties of a physical system L_1 (which is a "representation of a formal system") ? We obtain a theory $P = L_2 + S$ about L_1, that is, a formal system L_2 with a semantics S relating the elements of L_2 to the important empirical facts of L_1. That is, **instead of an "abstract structure"** we obtain another **flesh and blood** formal system L_2.

By the same token, one cannot obtain an "abstract structure" as an "equivalence class of isomorphic formal systems". **Such things as "isomorphism", "equivalence", "equivalence class" are living in a formal system "represented somehow"**, that is, **in a flesh and blood formal system**:



This is no attack on scientific realism When a *physical theory* claims that a physical object has a certain property adequately described by means of a formal system, then **this reflects a real feature of physical reality**.

This is not nominalism When many different physical objects display a similar property that is describable by means of the same (equivalent) elements of one common formal system, this will be a true *general* feature of the group.

But, **this realist commitment does not entitle us to claim that "abstract structures" exist *over and above* the real formal systems of physical existence**.

Epistemological status of meta-mathematical theories

We follow **Hilbert's** careful distinction:

mathematics – a system of meaningless signs

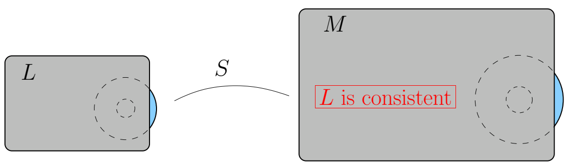
meta-mathematics – meaningful statements *about* mathematics

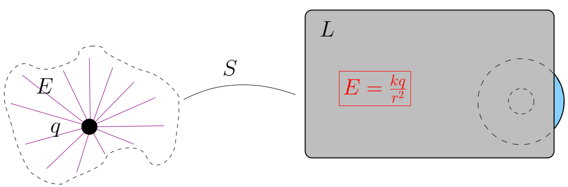
+ physicalism:

formal system – a physical system L

meta-mathematical theory – a *physical* theory (M, S)

All the truths that a meta-mathematical theory can tell us about its object are of the type **Truth**$_2$. This means that **no feature of a formal system can be "proved" mathematically: Genuine information about a formal system must be acquired by *a posteriori* means, that is, by observation of the formal system and, as in physics in general, by inductive generalization.**





Consequently, all meta-mathematical "proofs" are questionable!

- When I say "questionable" I do not mean that I don't believe that, for example, the sentence calculus is consistent. I only mean that **I believe in it just as I believe in the Coulomb law or in the conservation of energy, or any other physical laws, which are acquired by *a posteriori* means.**

- To be sure, both truth$_1$ and truth$_2$ of a formula of M, like

  L is consistent

  are known by *a posteriori* means. But, the *mathematical* theorem we derive in M,

  $\vdash_M$ L is consistent

  is known by observation of the formal system M, while the truth$_2$ of the meta-mathematical statement 'L is consistent' is confirmed by observations the object formal system L.

Part II

Weak points of Gödel's proof

Physicalist-formalist's reservations

Consider the following meta-mathematical statements:

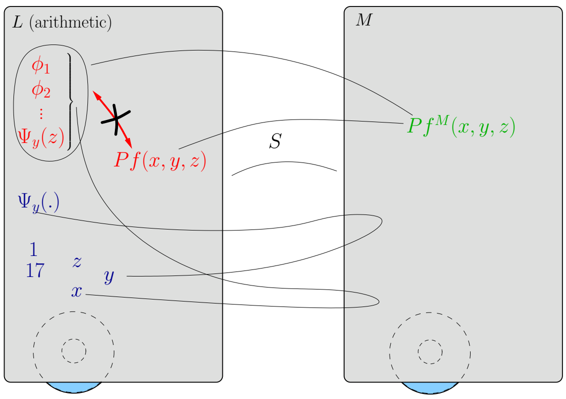
$Pf^M(x,y)$ — x is the Gödel number of a sequence of formulas constituting a proof of the formula of Gödel number y.

$Pf^M(x,y,z)$ — x is the Gödel number of a proof of the formula obtained from the formula of Gödel number y by substituting its only free variable with number z.

Representation:

$$\{\text{arithmetic}\} \vdash Pf(x,y,z) \text{ if and only if } Pf^M(x,y,z) \text{ is true}_2 \quad (2)$$

Problem 1 (2) is not "proved". It is known by *a posteriori* means!



Consider the following formula of arithmetic: $\neg\exists x Pf(x,y,y)$. Let g denote its Gödel number. The following formula of arithmetic is called Gödel sentence:

$$G \stackrel{def}{=} \neg\exists x Pf(x,g,g)$$

Theorem 1 *Neither G, nor $\neg G$ can be proved in arithmetic:*

$$\{arithmetic\} \nvdash G$$
$$\{arithmetic\} \nvdash \neg G$$

Problem 2 Since we use (2) in the proof, this is not a strict mathematical proof, but only an (although convincing) argument based on observations of the object formal system.

Formalist's reservations

It is usually claimed that the representation ϑ generated by Gödel's numbering is such that meta-mathematical statement 'G is not provable in arithmetic' is represented just by G. **But, what does "representation" mean?** Just like in (2), a meta-mathematical sentence A is represented by an arithmetical formula $\vartheta(A)$, when

$$A \text{ is true}_2 \text{ if and only if } \{\text{arithmetic}\} \vdash \vartheta(A)$$

Problem 3 According to this definition, however, **the Gödel sentence G does not represent the meta-mathematical sentence 'G is not provable in arithmetic'**! The representational property of ϑ does not extend to 'G is not provable in arithmetic':

> **NOT TRUE:** 'G is not provable in arithmetic' is true$_2$ iff $\{\text{arithmetic}\} \vdash \vartheta(\text{'}G \text{ is not provable in arithmetic'}) = G$

Finally, consider the following formula of arithmetic:

$$Consis \stackrel{def}{=} \forall x \neg Pf(x,k)$$

where k is the Gödel number of the formula $0 = 1$.

One can prove, in arithmetic, that $Consis \rightarrow G$. Consequently, disregarding the physicalist's delicate reservations against the proof of Gödel's first incompleteness theorem, one finds that $Consis$ is not provable in arithmetic.

It is, however, widely believed that $Consis$ is a sentence of arithmetic that represents the meta-mathematical statement 'Arithmetic is consistent'.

Problem 4 **This is, however, not the case!** According to the definition of representation, $Consis$ should satisfy the following condition:

$$\text{'Arithmetic is consistent' is true}_2 \text{ if and only if } \{\text{arithmetic}\} \vdash Consis$$

But, if arithmetic is consistent then $Consis$ is not a theorem, and, on the other hand, if arithmetic is inconsistent then $Consis$ is a theorem.